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Modulation of the Nonlinear Ion Acoustic Waves in A Weakly Relativistic Warm Plasma with Superthermally Distributed Electrons

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ABSTRACT

In this article we investigated the modulation of nonlinear ion acoustic waves in a weakly relativistic, warm, unmagnetized and adiabatic plasma whose constitutes are ion fluid and superthermally distributed electrons using the multiple scales approach. The basic system of equations is reduced to a finite wave number nonlinear Schrödinger-type equation at the second order of the perturbation theory and for small wave number limit the nonlinear Schrödinger-type equation is derived. Moreover the reductive perturbation technique is applied to this system and Korteweg-de Vries equation is obtained. For small wavenumber limit, it is found that the dispersion coefficient and nonlinear coefficient of the nonlinear Schrödinger-type equation are reduced to the coefficients of nonlinear Schrödinger-type equation obtained from Korteweg-de Vries (K-dV) equation. The dependence of the phase velocity and the group as well as the domain of the stability and the instability on the temperature ratio, the relativistic factor $\frac{u_0}{c}$ and the superthermal parameter is investigated.

Keywords:
Ion acoustic waves, Weakly relativistic, Superthermal electrons.

1. INTRODUCTION

One of the basic wave processes in plasma is the ion acoustic waves (IAWs), which has been studied for several decades both theoretically and experimentally.[1-7] The first observation of ion acoustic solitons has been reported experimentally by Ikezi et al.[8]. Also in 1977, Watanabe [9] reported the experimental observation of the modulation instability of the monochromatic IAWs.
The relativistic effect exists when the particle speed approaches that of light, but when the particle speed is much less than that of light, ion waves will exhibit nonrelativistic. El-Labany[10] and El-Labany et al.[11] have been studied theoretically the modulation instability of IAWs for different distribution by using the multiple scales method. Xue et al.[12] have used the reductive perturbation technique to study the modulation instability of IAWs in a warm plasma. The modulation instability of cold nonrelativistic behavior limits plasma has been studied for different distribution. The modulation instability has been

studied using nonthermally distributed electrons by Zhang et al.[13], q-nonextensively distributed electrons by Bains et al.[14] and kappa (κ) distribution by Guo and Mei[15] and Chowdhury et al.[16] .
The superthermal distribution describes the highly energetic particles which coexist in space and laboratory plasmas and their characteristics are deviated from the Maxwellian distribution. Sometimes these particles may be described by Lorentzian or κ distribution which also known as superthermal distribution [17-18]. Vasyliunas [19] introduced this distribution and its relation to Maxwellian distribution. The κ distribution may arise due to the external forces acting on a wave particle interaction or the neutral space plasmas. In fact this distribution goes back to Maxwellian distribution for limit of large spectral index [18] i.e., $\kappa \to \infty$.
There are different approximation techniques for describing the nonlinear evolutions in plasmas. Such as the multiple scales method [10-11], derivative expansion method [20] and Krylov-Bogoliubov-Mitropolsky [21] method. These techniques describe the small deviations for system from the equilibrium state of the linear wave. In this work we use the multiple scales method because it's more general and dependence on removing the secular terms.
However, the system of a weakly relativistic unmagnatized warm adiabatic plasma consisting inertial ions fluid and superthermally (κ) distributed electrons has not been investigated; this is our motive of the present investigation.
This article organized as follows:
The basic governing equations of the model and derivation of the non-linear Schrodinger-type (NST) equation are presented in section 2. We derive the small wave number approximation Korteweg-de Vries (K-dV) equation in section 3. We transform the K-dV equation obtained in section 3 to the NST equation in section 4 and results and discussion in section 5. Finally we devoted section 6 to conclusion.

2. GOVERNING EQUATIONS AND DERIVATION OF THE NST EQUATION

Consider a simple model of adiabatic unmagnetized collisionless weakly relativistic plasma that contains warm ion species together with superthermally distributed electrons. The one-dimensional basic equations can be written in normalized form as

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\gamma u + 3\sigma n\frac{\partial n}{\partial x} + \frac{\partial \Phi}{\partial x} = 0, \quad (2)$$

$$\frac{\partial^2 \Phi}{\partial x^2} = n_e - n \quad (3)$$

$$n_e = \left(1 - \frac{\Phi}{\left(\kappa - \frac{3}{2}\right)}\right)^{-\kappa+\frac{1}{2}} \approx 1 + \eta_1\Phi + \eta_2\Phi^2 + \eta_3\Phi^3, \quad (4)$$

where,

$$\left.\begin{aligned} \eta_1 &= \frac{\left(\kappa-\frac{1}{2}\right)}{\left(\kappa-\frac{3}{2}\right)}, \\ \eta_2 &= \frac{\left(\kappa-\frac{1}{2}\right)\left(\kappa+\frac{1}{2}\right)}{2\left(\kappa-\frac{3}{2}\right)^2}, \\ \eta_3 &= \frac{\left(\kappa-\frac{1}{2}\right)\left(\kappa+\frac{1}{2}\right)\left(\kappa+\frac{3}{2}\right)}{6\left(\kappa-\frac{3}{2}\right)^3}. \end{aligned}\right\}. \tag{5}$$

n , n_e are the numbers densities of the ions and electrons normalized by unperturbed ion density n_0, u is the flow speed of the ions normalized by thermal velocity $(k_B T_e/m)^{\frac{1}{2}}$, Φ is the electrostatic potential normalized by thermal potential $(k_B T_e/e)$, x is the space coordinate normalized by Debye length $\lambda_D = (k_B T_e/4\pi e^2 n_0)^{\frac{1}{2}}$, t is the time variable normalized by the inverse of the ion plasma frequency $\omega_{pi}^{-1} = (4\pi e^2 n_0/m)^{\frac{1}{2}}$. $\sigma \ll 1$ is the ratio of ion temperature Ti to electron temperature Te and the parameter κ stands for the strength of superthermally, where m is the ion mass, k_B is the Boltzmann constant and e is the electron charge. For a weakly relativistic effect, the relativistic factor γ is approximated by (Gill et al.[23])

$$\gamma = \left(1-\frac{u^2}{c^2}\right)^{\frac{-1}{2}} \approx 1 + \frac{\mathrm{u}^2}{2\mathrm{c}^2} \tag{6}$$

To obtain the NST equation, we employ the general method of multiple scales. In this method we introduce the stretched variables (El-labany et al.) [11]

$$\left.\begin{aligned} \tau_i = \varepsilon^i t, \xi_0 = x, \\ \xi_i = \varepsilon^i\,(x-\lambda t) \quad (i = 1,2,\dots\dots). \end{aligned}\right\} \tag{7a}$$

Using this method, the time and space derivatives in Eqs (1-4) are written as

$$\left.\begin{aligned} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau_0} + \varepsilon\left(\frac{\partial}{\partial \tau_1} - \lambda\frac{\partial}{\partial \xi_1}\right) + \varepsilon^2\left(\frac{\partial}{\partial \tau_2} - \lambda\frac{\partial}{\partial \xi_2}\right) + \cdots, \\ \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \xi_0} + \varepsilon\frac{\partial}{\partial \xi_1} + \varepsilon^2\frac{\partial}{\partial \xi_2} + \cdots, \end{aligned}\right\} \tag{7b}$$

where ε is a small dimensionless parameter characterizes the size of the perturbed amplitude and λ characterizes the group velocity ($\lambda = \frac{\partial \omega}{\partial k}$); will be determined later. We expand the dependent variables n, u and Φ in terms of the expansion parameter ε as (EL-Labany & EL-Labany et al.) [10-11]

$$\begin{pmatrix} n \\ u \\ \Phi \end{pmatrix} = \begin{pmatrix} 1 \\ u_0 \\ 0 \end{pmatrix} + \sum_{m=1}^{\infty}\sum_{l=\infty}^{-\infty} \varepsilon^m \begin{pmatrix} n_m^{(l)}(\tau_1,\tau_2,\dots.,\xi_1,\xi_2,\dots.) \\ u_m^{(l)}(\tau_1,\tau_2,\dots.,\xi_1,\xi_2,\dots.) \\ \Phi_m^{(l)}(\tau_1,\tau_2,\dots.,\xi_1,\xi_2,\dots.) \end{pmatrix} e^{il(kx-\omega t)} \tag{7c}$$

where n, u and Φ are satisfied the reality condition $A_{-l}^{(m)} = A_l^{(m)*}$, where the asterisk denotes the complex conjugate.

Using Eqs. (7) into the basic equations (1)-(4), we obtain to the first order of ε and $l = 1$ components as

$$\left.\begin{array}{c} u_1^{(1)} = \dfrac{\widetilde{\omega}}{k} n_1^{(1)}, \\ and \\ \Phi_1^{(1)} = \dfrac{n_1^{(1)}}{(k^2 + \eta_1)}. \end{array}\right\} \tag{8}$$

Then, the linear dispersion relation and group velocity λ are given respectively by

$$\widetilde{\omega}^2 \gamma_1 = 3\sigma k^2 + \frac{k^2}{k^2 + \eta_1}, \tag{9a}$$

$$\lambda = u_0 + \frac{k}{\gamma_1 \widetilde{\omega}}\left(3\sigma + \frac{\eta_1}{(k^2 + \eta_1)^2}\right), \tag{9b}$$

where, $\gamma_1 = 1 + \frac{3{u_0}^2}{2C^2}$ and $\widetilde{\omega} = \omega - k u_0$.

The components of O(ε) for $l = 0$ are

$$\left.\begin{array}{c} n_1^{(0)} = n_{e1}^{(0)}, \\ and \\ \Phi_1^{(0)} = \dfrac{n_1^{(0)}}{\eta_1}, \end{array}\right\} \tag{10}$$

The second order harmonic O(ε^2) with $l = 0$ are given by,

$$\left.\begin{array}{c} \dfrac{\partial n_1^{(0)}}{\partial \xi_1} = \dfrac{\partial u_1^{(0)}}{\partial \xi_1} = \dfrac{\partial \Phi_1^{(0)}}{\partial \xi_1} = 0, \\ \Phi_1^{(0)} = 0, \\ and, \\ \Phi_2^{(0)} = \dfrac{\left(n_2^{(0)} - 2\eta_2 \left|\Phi_1^{(1)}\right|^2\right)}{\eta_1}, \\ provided\ that \\ \gamma_1 \tilde{\lambda}^2 \neq \dfrac{1}{\eta_1} + 3\sigma; \end{array}\right\} \tag{11}$$

and for $l = 1$ components,

$$\left.\begin{array}{c}\dfrac{\partial n_1^{(1)}}{\partial \tau_1} = 0,\\[2mm] u_2^{(1)} = \dfrac{\widetilde{\omega}}{k} n_2^{(1)} + \dfrac{\mathrm{i}}{k}\left(\dfrac{\widetilde{\omega}}{k} - \tilde{\lambda}\right)\dfrac{\partial n_1^{(1)}}{\partial \xi_1},\\ and \\ \Phi_2^{(1)} = \dfrac{n_2^{(1)}}{(k^2+\eta_1)} + \dfrac{2\mathrm{i}k}{(k^2+\eta_1)^2}\dfrac{\partial n_1^{(1)}}{\partial \xi_1};\end{array}\right\} \quad (12)$$

i.e. no τ_1 dependent.

For $l = 2$ components,

$$\begin{pmatrix} n_2^{(2)} \\ u_2^{(2)} \\ \Phi_2^{(2)} \end{pmatrix} = \begin{pmatrix} A_n \\ A_u \\ A_\Phi \end{pmatrix} n_1^{(1)^2}, \quad (13)$$

where,

$$A_n = (k^2 + \eta_1)\left\{\frac{\widetilde{\omega}^2}{k^2}\left(\frac{3}{2}\gamma_1 - \frac{\widetilde{\omega}}{k}\gamma_2\right) + \frac{3}{2}\sigma + A_\Phi\right\},$$

$$A_u = \frac{\widetilde{\omega}}{k}(A_n - 1),$$

$$A_\Phi = \frac{(k^2+\eta_1)}{3k^2}\left\{\frac{\widetilde{\omega}^2}{k^2}\left(\frac{3}{2}\gamma_1 - \frac{\widetilde{\omega}}{k}\gamma_2\right) + \frac{3}{2}\sigma - \frac{\eta_2}{(k^2+\eta_1)^3}\right\},$$

$$\widetilde{\lambda} = \lambda - u_0,$$

and,

$$\gamma_2 = \frac{3u_0}{2C^2};$$

However, the second-order quantities with zeroth harmonic are determined from third order; O(ε^3) and are given by,

$$\begin{pmatrix} n_2^{(0)} \\ u_2^{(0)} \\ \Phi_2^{(0)} \end{pmatrix} = \begin{pmatrix} B_n \\ B_u \\ B_\Phi \end{pmatrix} \left|n_1^{(1)}\right|^2, \quad (14)$$

where,

$$B_n = \frac{1}{\tilde{\lambda}}\left(\frac{2\widetilde{\omega}}{k} + B_u\right),$$

$$B_u = \frac{1}{z}\left\{\frac{\widetilde{\omega}^2\tilde{\lambda}^2}{k^2}\left(\frac{\gamma_1}{\tilde{\lambda}} - 2\gamma_2\right) + 3\sigma\left(\tilde{\lambda} + \frac{2\widetilde{\omega}}{k}\right) + \frac{2\widetilde{\omega}}{\eta_1 k} - \frac{2\eta_2\tilde{\lambda}}{\eta_1(k^2+\eta_1)^2}\right\},$$

$$B_{\Phi} = \frac{1}{\eta_1}\left\{\frac{B_n(k^2+\eta_1)^2 - 2\eta_2}{(k^2+\eta_1)^2}\right\},$$
$$and,$$
$$z = \gamma_1\tilde{\lambda}^2 - 3\sigma - \frac{1}{\eta_1}.$$

Using the above mentioned quantities and O(ε^3) with $l = 1$ we get the NST equation,

$$\mathrm{i}\frac{\partial n_1^{(1)}}{\partial \tau} + P\frac{\partial^2 n_1^{(1)}}{\partial {\xi_1}^2} + Q\, n_1^{(1)}\left|n_1^{(1)}\right|^2 = 0. \quad (15)$$

where,

$$P = \frac{-k^2}{2\tilde{\omega}\gamma_1(k^2+\eta_1)^3}\left\{-(k^2-3\eta_1) + \frac{(k^2+\eta_1)^3}{k^2}\left(\frac{\tilde{\omega}^2\gamma_1}{k^2} - \frac{2\tilde{\omega}\gamma_1\tilde{\lambda}}{k} + \gamma_1\tilde{\lambda}^2\right)\right\} = \frac{1}{2}\frac{\partial^2\tilde{\omega}}{\partial k^2}$$

and,

$$Q = \frac{-k^2}{2\tilde{\omega}\gamma_1}\left\{\left(\frac{\tilde{\omega}^2}{k^2}\gamma_1 + 3\sigma\right)(A_n + B_n) + \frac{2\tilde{\omega}}{k}\left(\gamma_1 - \frac{\tilde{\omega}}{k}\gamma_2\right)(A_u + B_u) - \frac{2\eta_2}{(k^2+\eta_1)^2}(A_{\Phi} + B_{\Phi}) + 2\left(\frac{\tilde{\omega}}{k}\right)^3\gamma_2 - \frac{3}{2C^2}\left(\frac{\tilde{\omega}}{k}\right)^4 - \frac{3\eta_3}{(k^2+\eta_1)^4}\right\}.$$

For finite wave number region, equation (15) satisfies the evolution of the complex amplitude of the nonlinear ion acoustic waves (IAWs), propagating in a weakly relativistic warm with superthermally distributed electrons on the basis of the fluid model.

For small wave number, equation (15) reduces to

$$\mathrm{i}\frac{\partial n_1^{(1)}}{\partial \tau} - \frac{3}{2}\frac{bk}{\eta_1^2}\frac{\partial^2 n_1^{(1)}}{\partial \xi^2} + \frac{1}{3k}\,\frac{a^2\eta_1^2}{b}n_1^{(1)}\left|n_1^{(1)}\right|^2 = 0, \quad (16)$$

where,

$$a = \left[\frac{\gamma_1(3\sigma\eta_1+1)}{\eta_1}\right]^{\frac{-1}{2}}\left\{\frac{(3\sigma\eta_1+1)(3\eta_1^2-2\eta_2)}{2\eta_1^3} + \frac{3}{2}\frac{\sigma(\eta_1^2+2\eta_2)}{\eta_1^2} - \frac{\gamma_2}{\gamma_1^{\frac{3}{2}}}\left(\frac{3\sigma\eta_1+1}{\eta_1}\right)^{\frac{3}{2}}\right\},$$

and,

$$b = \left[\frac{\gamma_1(3\sigma\eta_1+1)}{\eta_1}\right]^{\frac{-1}{2}}.$$

3. DERIVATION K-DV EQUATION FOR THE SYSTEM

We apply the reductive perturbation theory (Nejoh[2]), to show that the amplitude of the perturbed ion density in a weakly relativistic warm plasma and superthermally distributed electrons in the small-wavenumber limit is governed by the K-dV equation. To show that we introduce the stretched variables ξ and τ as (Pakzad and Gill et al.[22-23])

$$\xi = \mu^{\frac{1}{2}}(x - \bar{\lambda}t), \;\; \tau = \mu^{\frac{3}{2}}\,t \quad (17a)$$

Thus,

$$\left.\begin{aligned}\frac{\partial}{\partial x} &= \mu^{\frac{1}{2}}\frac{\partial}{\partial \xi},\\ \frac{\partial}{\partial t} &= \mu^{\frac{1}{2}}(-\bar{\lambda}\frac{\partial}{\partial \xi} + \mu\frac{\partial}{\partial \tau})\end{aligned}\right\} \tag{17b}$$

and we expand the dependent variables n, u, and Φ in μ as

$$\left.\begin{aligned}n &= 1 + \mu\tilde{n}_1 + \mu^2\tilde{n}_2 + \mu^3\tilde{n}_3 + \cdots\\ u &= u_0 + \mu u_1 + \mu^2 u_2 + \mu^3 u_3 + \cdots\\ \Phi &= \mu\Phi_1 + \mu^2\Phi_2 + \mu^3\Phi_3 + \cdots\end{aligned}\right\} \tag{17c}$$

where $\tilde{n}$ is the perturbed ion density and μ is the ordering parameter and is a measure of the size of the wavenumber k; that is, $k = O\left(\mu^{\frac{1}{2}}\right)$.

Introducing Eqs. (17) into the basic set of Eqs. (1)-(3) and equating the similar power coefficients, to the lowest order terms of μ we have

$$\left.\begin{aligned}\tilde{n}_1 &= \eta_1\Phi_1,\\ u_1 &= \grave{\lambda}\tilde{n}_1,\\ &and\\ \Phi_1 &= \frac{\tilde{n}_1}{\eta_1},\end{aligned}\right\} \tag{18a}$$

where,

$$\grave{\lambda} = \bar{\lambda} - u_0 \tag{18b}$$

The Poisson's equation, gives the compatibility condition

$$(\grave{\lambda}^2\gamma_1 - 3\sigma)\eta_1 = 1 \tag{18c}$$

The next order of μ gives,

$$\frac{\partial \tilde{n}_1}{\partial \tau} - (\bar{\lambda} - u_0)\frac{\partial \tilde{n}_2}{\partial \xi} + \frac{\partial u_2}{\partial \xi} + \grave{\lambda}\frac{\partial \tilde{n}_1^2}{\partial \xi} = 0\,, \tag{19a}$$

$$\begin{gathered}\gamma_1\grave{\lambda}\frac{\partial \tilde{n}_1}{\partial \tau} - (\bar{\lambda} - u_0)\frac{\partial}{\partial \xi}\left(\gamma_1 u_2 + \gamma_2\grave{\lambda}^2{\tilde{n}_1}^2\right) + \gamma_1\grave{\lambda}^2\tilde{n}_1\frac{\partial \tilde{n}_1}{\partial \xi}\\ + 3\sigma\left[\frac{\partial \tilde{n}_2}{\partial \xi} + \tilde{n}_1\frac{\partial \tilde{n}_1}{\partial \xi}\right] + \frac{\partial \Phi_2}{\partial \xi} = 0,\end{gathered} \tag{19b}$$

and

$$\frac{\partial^2\Phi_1}{\partial \xi^2} = \eta_1\Phi_2 + \eta_2\frac{\tilde{n}_1^2}{\eta_1^2} - \tilde{n}_2\,. \tag{19c}$$

Using the results of the pervious order and eliminating the second order perturbed quantities with some algebraic manipulations we obtain the KdV equation,

$$\frac{\partial \tilde{n}}{\partial \tau} + a\tilde{n}\frac{\partial \tilde{n}}{\partial \xi} + \frac{b}{2{\eta_1}^2}\frac{\partial^3\tilde{n}}{\partial \xi^3} = 0, \tag{20}$$

where a and b are given by

$$a = \left[\frac{\gamma_1(3\sigma\eta_1 + 1)}{\eta_1}\right]^{\frac{-1}{2}}\left\{\frac{(3\sigma\eta_1 + 1)(3\eta_1^2 - 2\eta_2)}{2\eta_1^3} + \frac{3}{2}\frac{\sigma(\eta_1^2 + 2\eta_2)}{\eta_1^2} - \frac{\gamma_2}{\gamma_1^{\frac{3}{2}}}\left(\frac{3\sigma\eta_1 + 1}{\eta_1}\right)^{\frac{3}{2}}\right\},$$

and,

$$b = \left[\frac{\gamma_1(3\sigma\eta_1 + 1)}{\eta_1}\right]^{\frac{-1}{2}}.$$

4. Derivation of NST equation from K-dV equation (small wave number approximation)

To derive the NST equation from K-dV equation (20) we follow El-Labany[10] and El-Labany, et al. [11] we expand $\tilde{n}$ as

$$\tilde{n} = \sum_{m=1}^{\infty} \tilde{n}_l^{(m)}(\chi, \sigma) \exp[\mathrm{i}l(k\xi - \Omega\tau)], \tag{21}$$

where the stretched variables $\chi\ and\ \eta$ are related to $\xi\ and\ \tau$ by

$$\zeta = \varepsilon^2\tau, \qquad \chi = \varepsilon(\xi - \delta\tau)$$

δ and Ω will be determined later and the time and space derivatives in (20) are expressed by (El-Labany & El-Hanbaly [24])

$$\frac{\partial}{\partial\tau} \to \frac{\partial}{\partial\tau} - \varepsilon\delta\frac{\partial}{\partial\chi} + \varepsilon^2\frac{\partial}{\partial\zeta}, \tag{22a}$$

$$\frac{\partial}{\partial\xi} \to \frac{\partial}{\partial\xi} + \varepsilon\frac{\partial}{\partial\chi}. \tag{22b}$$

Using (21) and (22) into (20), then the reduced equation of order m is written as:

$$\begin{aligned} -\mathrm{i}l\Omega\tilde{n}_l^{(m)} - \delta\frac{\partial\tilde{n}_l^{(m-1)}}{\partial\chi} + \frac{\partial\tilde{n}_l^{(m-2)}}{\partial\zeta} - \frac{\mathrm{i}b}{2{\eta_1}^2}(kl)^3\tilde{n}_l^{(m)} - \frac{3b}{2{\eta_1}^2}(kl)^2\frac{\partial\tilde{n}_l^{(m-1)}}{\partial\chi} \\ + \frac{3\mathrm{i}b}{2{\eta_1}^2}kl\frac{\partial^2\tilde{n}_l^{(m-2)}}{\partial\chi^2} + \frac{b}{2{\eta_1}^2}\frac{\partial^3\tilde{n}_l^{(m-3)}}{\partial\chi^3} \\ + \mathrm{i}ak\sum_{\acute{m}}^{m-1}\sum_{\acute{l}=-\infty}^{\infty}(l-\acute{l})\tilde{n}_{\acute{l}}^{(\acute{m})}\tilde{n}_{l-\acute{l}}^{(m-\acute{m})} \\ + a\sum_{\ddot{m}}^{m-2}\sum_{\ddot{l}=-\infty}^{\infty}\tilde{n}_{\ddot{l}}^{(\ddot{m})}\frac{\partial\tilde{n}_{l-\ddot{l}}^{(m-\ddot{m}-1)}}{\partial\chi} = 0, \end{aligned} \tag{23}$$

The first-order terms (m=1) with $l = \pm1$ from equation (23) gives

$$\left(\Omega + \frac{1}{2{\eta_1}^2}k^3\right)\tilde{n}_{\pm1}^{(1)} = 0,$$

and the linear dispersion relation is given by

$$\Omega = \frac{-1}{2{\eta_1}^2}bk^3.$$

From equation (22), the Second-harmonic components of the second-order terms (m=2) is given as

$$\tilde{n}_2^{(2)} = \frac{{\eta_1}^2}{3k^2}\frac{a}{b}\tilde{n}_1^{(1)}\tilde{n}_1^{(1)},$$

and the $l = \pm1$ components of this order lead to the equation

$$\left(\delta + \frac{3}{2{\eta_1}^2}bk^2\right)\frac{\partial\tilde{n}_{\pm1}^{(1)}}{\partial\chi} = 0,$$

Then, the compatibility condition for non-trivial solution is written as

$$\delta = -\frac{3}{2{\eta_1}^2}bk^2.$$

The zeroth-harmonic components of the third-order terms (m=3) mth reduced equation are written as

$$\tilde{n}_2^{(0)} = \frac{a}{\delta}\left|\tilde{n}_1^{(1)}\right|^2,$$

Finally, from the third-order terms of the reduced equation for $l = 1$ we obtain the NST equation

$$\mathrm{i}\frac{\partial n_1^{(1)}}{\partial \zeta} - \frac{3}{2}\frac{bk}{\eta_1^2}\frac{\partial^2 n_1^{(1)}}{\partial \chi^2} + \frac{1}{3k}\frac{a^2\eta_1^2}{b} n_1^{(1)}\left|n_1^{(1)}\right|^2 = 0. \quad (24)$$

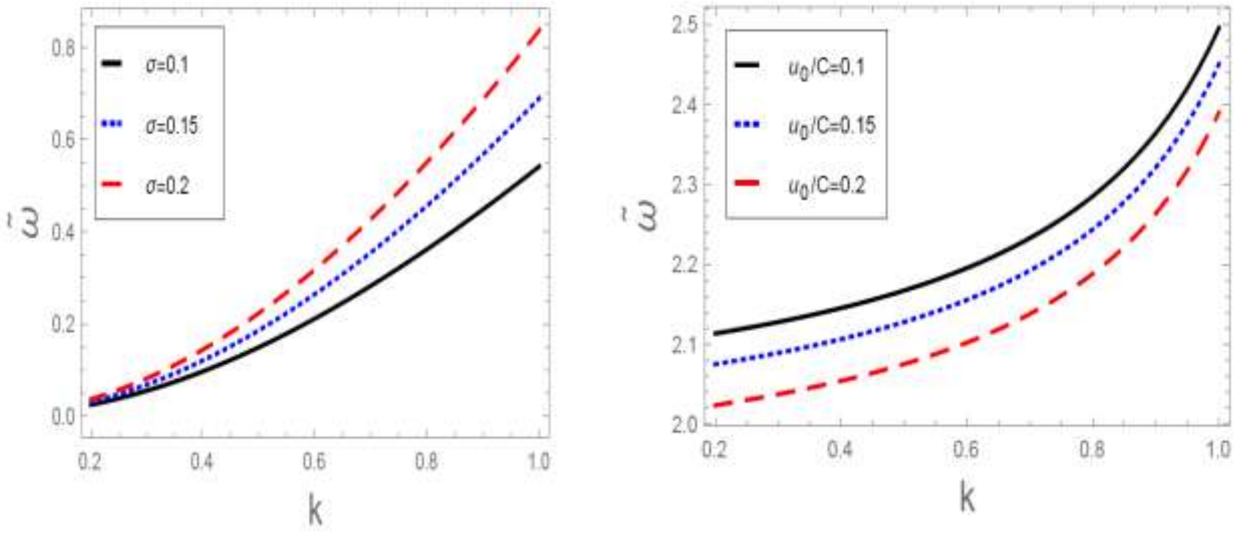
The same as those obtained from the small wavenumber limit of equation (15) i.e. equation (16).

4. RESULTS AND DISCUSSION

We have numerically analyzed Eq. (9a) to examine the linear properties of the IAWs for different values of σ and $\frac{u_0}{c}$ (see figure 1 a and b). It is obvious from these figures that the phase velocity increases with increasing σ. Unlike the temperature ratio (σ), when the velocity ratio $\left(\frac{u_0}{c}\right)$ increases the phase velocity decreases. Also, we examine numerically the group velocity (see Eq. 9b) properties of IAWs for different values of σ and $\frac{u_0}{c}$ (see figure 2). The group velocity decreases with increasing superthermal parameter κ, but increases with increasing σ and independent on increasing $\frac{u_0}{c}$.

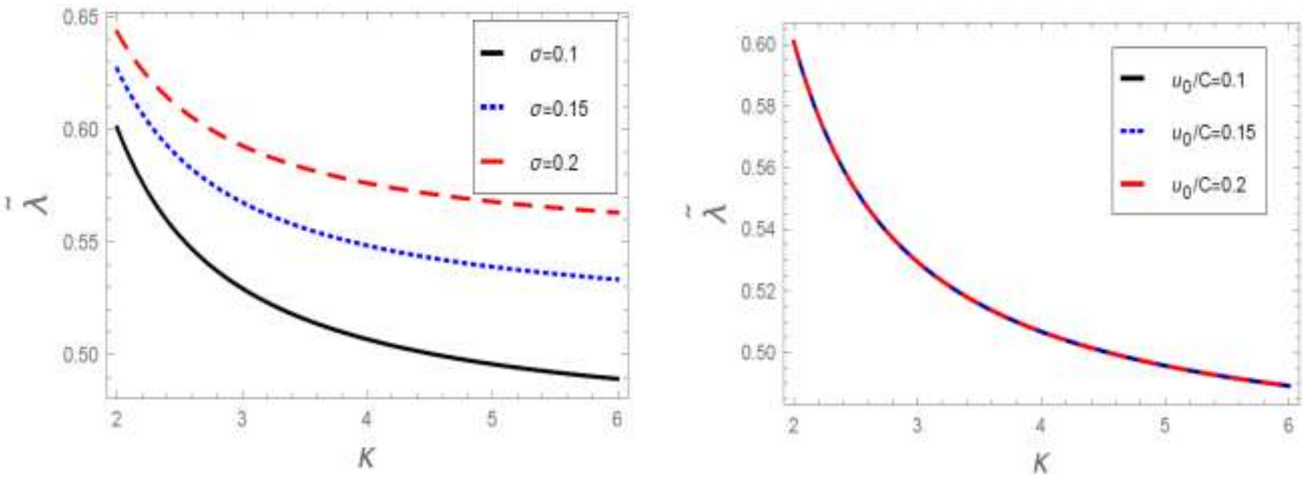
Now, we investigate the stability and instability regions of the ion acoustic waves for our plasma model on the basis of the NST equation (15). Based on the stability analysis and the importance of PQ sign,[16,25] if PQ < 0, the amplitude of the modulated wave will be "stable" and if PQ > 0, the amplitude of the modulated wave will be "unstable". Two regions of stable and unstable give two stationary solutions: Stable solutions called dark envelope soliton for "negative" PQ, and unstable solutions called bright envelope soliton for "positive" PQ. According to this analysis, the modulation instability of IAWs has been studied. Figure 3 shows the variation of the critical wave numbers (higher and lower wave number) with $\frac{u_0}{c}$ for different σ. It is clear from these figures that the upper and lower critical wavenumber decrease with increasing σ. We note that that when the temperature ratio increase the stable region decrease as the phase and the group velocities increase with increase σ, thus the system becomes more unstable. Which, the bright and yellow color represented the stability and instability regions respectively. Therefore, the rouge waves may propagate for plasma parameters within the unstable region (PQ > 0). Figure 4 shows the variation of the critical wave numbers with σ for different superthermal parameter κ, the upper and lower critical wavenumbers also decrease with increasing superthermal parameter κ. Because the superthermally distributed electrons contain a large number of electrons that have high energy, which cause instability in the system and thus reduce the stable areas of the system. Then, when the kappa parameter increases $(\kappa \to \infty)$ the superthermal distribution goes back to Maxwellian distribution as shown in Figs. 5 and 6.

Finally, we note that the system is strongly dependent on the superthermal parameter κ and this is important for studding the highly energetic particles which coexist in space plasmas. For examples, superthermal electrons have been observed in the Earth's bow-shock,[26] and in the magnetospheres of Jupiter and Saturn.[27-28]



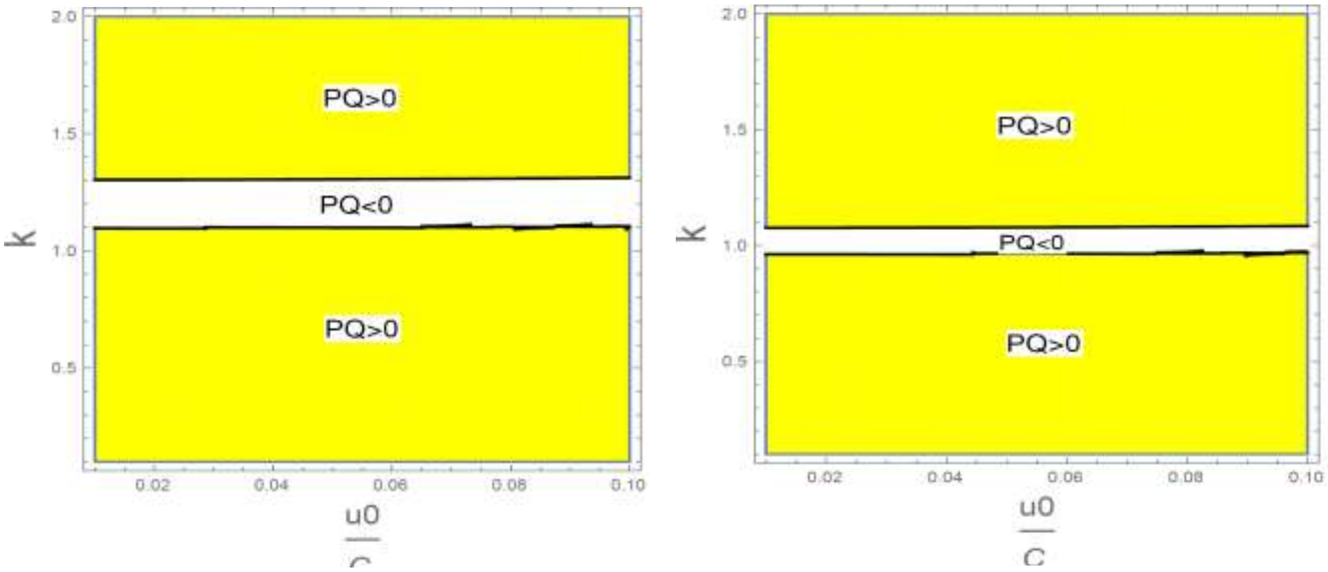
(a) (b)

Fig. 1.The variation of $\tilde{\omega}$ (wave frequency) with wavenumber k: (a) for different values of σ and $\frac{u_0}{C} = 0.1$, (b) for different values of $^{u_0}/_C$ and $\sigma = 0.1$. Here, the superthermal parameter $\varkappa = 2$.



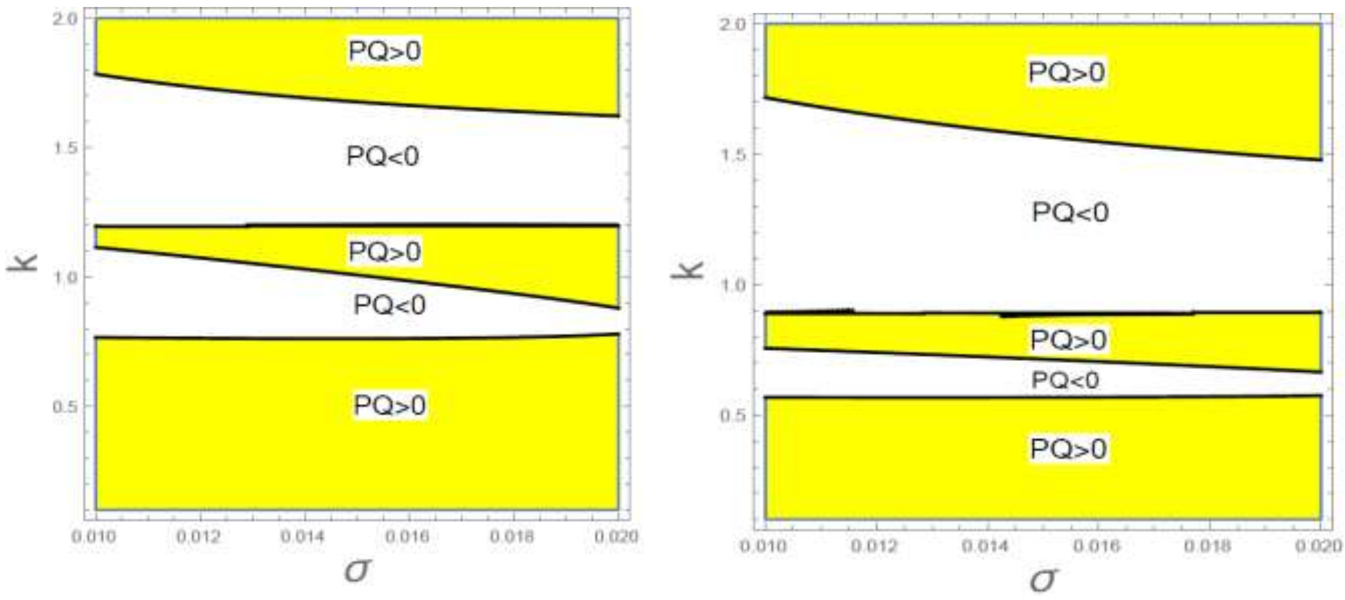
(a) (b)

Fig. 2.The variation of $\tilde{\lambda}$ (group velocity) with superthermal parameter κ: (a) for different values of σ and $\frac{u_0}{C} = 0.2$, (b) for different values of $\frac{u_0}{C}$ and $\sigma = 0.1$. Here, the wavenumber $k = 1.4$.



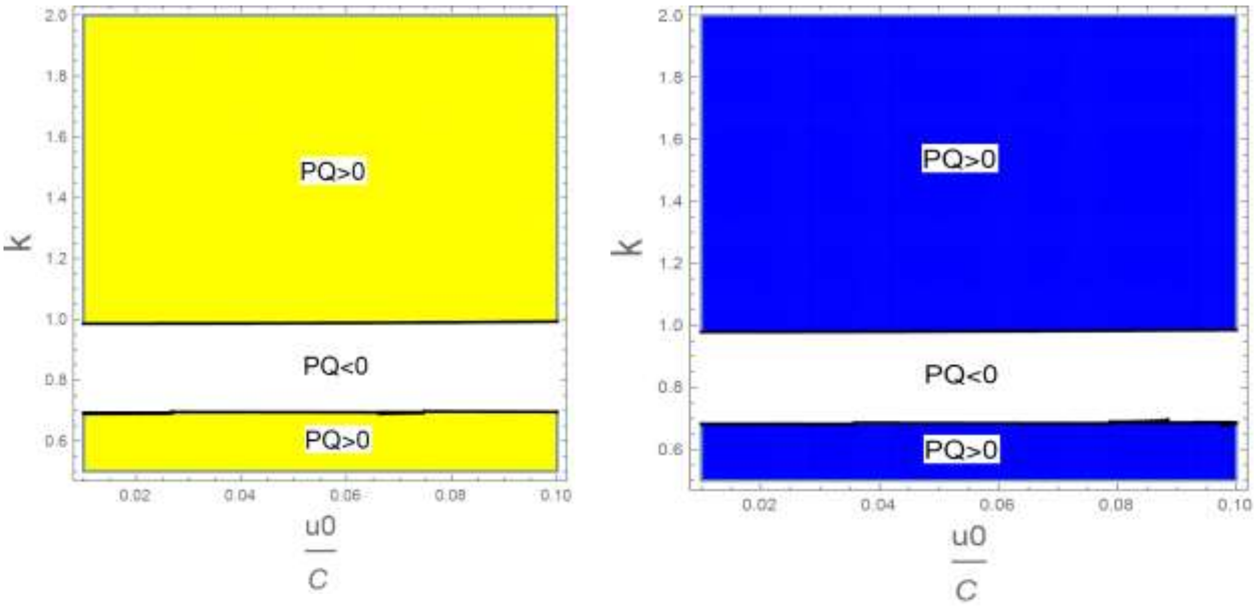
(a) (b)

Fig. 3. The contour plot of the product PQ = 0, against k and $\frac{u_0}{C}$: (a) for $\sigma = 0.1$, (b) for $\sigma = 0.2$ .Here, the superthermal parameter κ =2, where the (bright) yellow region represents the region in which (stability) instability.



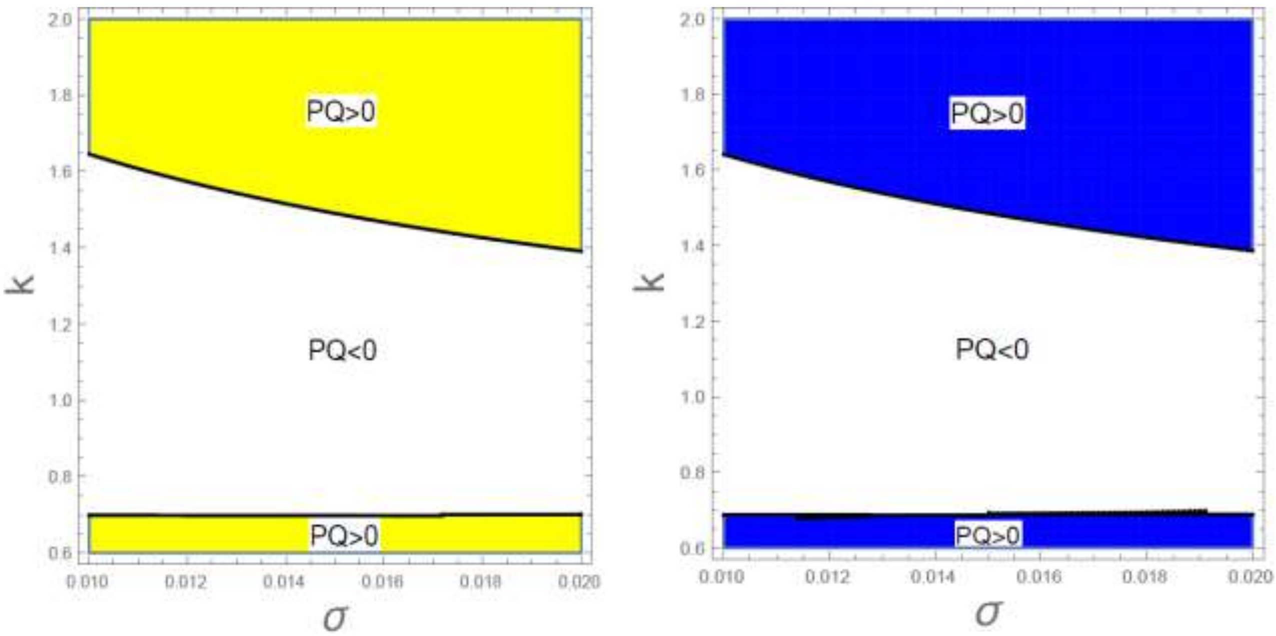
(a) (b)

Fig. 4. The contour plot of the product PQ = 0, against k and σ : (a) for the superthermal parameter κ =2, (b) for the superthermal parameter κ =3. Here, $\frac{u_0}{C} = 0.1$, where the (bright) yellow region represents the region in which (stability) instability.



(a) (b)

Fig. 5. The contour plot of the product PQ = 0, against k and $\frac{u_0}{C}$: (a) for large superthermal parameter κ =35, (b) for Maxwellian distribution .Here, $\sigma = 0.1$ and $\frac{u_0}{C} = 0.1$, where the (bright) blue region represents the region in which (stability) instability.



(a) (b)

Fig. 6. The contour plot of the product PQ = 0, against k and σ: (a) for large super thermal parameter κ =35, (b) for Maxwellian distribution. Here, $\sigma = 0.1$ and $\frac{u_0}{C} = 0.1$, where the (bright) blue region represents the region in which (stability) instability.

5. CONCLUSION

In this article, we investigated the modulation of the nonlinear acoustic wave in a warm unmagnetized plasma consists of superthermal electrons and inertial adiabatic ions. Using the multiple scales approach a NST equation is found whose coefficients of the dispersion and nonlinear terms are dependent on σ and the streaming velocity through $\gamma_1$and γ_2. For finite amplitude we derived K-dV equation for this system, which is transformed into NST equation. The small wavenumber limit of the coefficients of the original NST equation agree with those NST equation obtained from K-dV. Moreover we obtained the effect of σ and u_0 as domain of the stability ($PQ < 0$) and the unstable region ($PQ > 0$) by determining the critical wave at which the sign of PQ changes from positive to negative and vice versa. Also we considered the case $\varkappa \rightarrow \infty$ to check the stability of our work. A good agreement is found with work done by El-Labany.[10] . Finally, The dependence of the phase velocity and the group as well as the domain of the stability and the instability on the physical parameters σ, $\frac{u_0}{c}$ and κ is investigated.

Appendix

To investigate the coefficients of the NST equation for small limit wavenumber (k), firstly we calculate the different terms appearing in these coefficients.

From Eqs.(9), as $k \rightarrow 0$, we have

$$\frac{\widetilde{\omega}}{k} \approx \check{\lambda} \approx \left(\frac{1 + 3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}},$$

thus

$$\left[\frac{1}{z}\right] = \left[\frac{\eta_1(k^2 + \eta_1)}{k^2}(k^2\eta_1\{3\gamma_1\sigma(k^2 + \eta_1) + 1\} - (k^2 + 3\eta_1)(k^2 + \eta_1))^{-1}\right] \approx \frac{-\eta_1^2}{3k^2},$$

where the square-bracket notation indicates the quantity is estimated at small wave number ($k \rightarrow 0$). Also

$$[A_n] = (k^2 + \eta_1)\left\{\frac{\widetilde{\omega}^2}{k^2}\left(\frac{3}{2}\gamma_1 - \frac{\widetilde{\omega}}{k}\gamma_2\right) + \frac{3}{2}\sigma + A_\Phi\right\}$$

$$\approx \frac{\eta_1^2}{3k^2}\left\{\frac{3}{2}\left(\frac{3\sigma\eta_1 + 1}{\eta_1}\right) - \frac{\gamma_2}{\gamma_1^{\frac{3}{2}}}\left(\frac{1 + 3\sigma\eta_1}{\eta_1}\right)^{\frac{3}{2}} + \frac{3}{2}\sigma - \frac{\eta_2}{\eta_1^3}\right\},$$

$$[\,A_u] = \frac{\widetilde{\omega}}{k}(A_n - 1) \approx \frac{\eta_1^2}{3k^2}\left(\frac{1 + 3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\left\{\frac{3}{2}\left(\frac{3\sigma\eta_1 + 1}{\eta_1}\right) - \frac{\gamma_2}{\gamma_1^{\frac{3}{2}}}\left(\frac{1 + 3\sigma\eta_1}{\eta_1}\right)^{\frac{3}{2}} + \frac{3}{2}\sigma - \frac{\eta_2}{\eta_1^3}\right\},$$

$$[A_\Phi] = \frac{(k^2 + \eta_1)}{3k^2}\left\{\frac{\widetilde{\omega}^2}{k^2}\left(\frac{3}{2}\gamma_1 - \frac{\widetilde{\omega}}{k}\gamma_2\right) + \frac{3}{2}\sigma - \frac{\eta_2}{(k^2 + \eta_1)^3}\right\}$$

$$\approx \frac{\eta_1}{3k^2}\left\{\frac{3}{2}\left(\frac{3\sigma\eta_1 + 1}{\eta_1}\right) - \frac{\gamma_2}{\gamma_1^{\frac{3}{2}}}\left(\frac{1 + 3\sigma\eta_1}{\eta_1}\right)^{\frac{3}{2}} + \frac{3}{2}\sigma - \frac{\eta_2}{\eta_1^3}\right\},$$

$$[B_n] = \frac{1}{\tilde{\lambda}}\left(\frac{2\widetilde{\omega}}{k} + \frac{1}{z}\left\{\frac{\widetilde{\omega}^2\tilde{\lambda}^2}{k^2}\left(\frac{\gamma_1}{\tilde{\lambda}} - 2\gamma_2\right) + 3\sigma\left(\tilde{\lambda} + \frac{2\widetilde{\omega}}{k}\right) + \frac{2\widetilde{\omega}}{\eta_1 k} - \frac{2\eta_2}{\eta_1(k^2+\eta_1)^2}\right\}\right)$$

$$\approx \frac{-\eta_1^2}{3k^2}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{-1}{2}}\left\{\gamma_1\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{3}{2}} - 2\gamma_2\left(\frac{1+3\sigma\eta_1}{\eta_1\gamma_1}\right)^{2} + 9\sigma\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\right.$$

$$\left. + \frac{2}{\eta_1}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}} - \frac{2\eta_2}{\eta_1^3}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\right\},$$

$$[B_u] = \frac{1}{z}\left\{\frac{\widetilde{\omega}^2\tilde{\lambda}^2}{k^2}\left(\frac{\gamma_1}{\check{\lambda}} - 2\gamma_2\right) + 3\sigma\left(\tilde{\lambda} + \frac{2\widetilde{\omega}}{k}\right) + \frac{2\widetilde{\omega}}{\eta_1 k} - \frac{2\eta_2\tilde{\lambda}}{\eta_1(k^2+\eta_1)^2}\right\}$$

$$\approx \frac{-\eta_1{}^2}{3k^2}\left\{\gamma_1\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{3}{2}} - 2\gamma_2\left(\frac{1+3\sigma\eta_1}{\eta_1\gamma_1}\right)^{2} + 9\sigma\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}} + \frac{2}{\eta_1}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\right.$$

$$\left. - \frac{2\eta_2}{\eta_1{}^3}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\right\},$$

and

$$[B_\Phi] = \frac{1}{\eta_1}\left\{\frac{B_n(k^2+\eta_1)^2 - 2\eta_2}{(k^2+\eta_1)^2}\right\}$$

$$\approx \frac{-\eta_1}{3k^2}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{-1}{2}}\left\{\gamma_1\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{3}{2}} - 2\gamma_2\left(\frac{1+3\sigma\eta_1}{\eta_1\gamma_1}\right)^{2} + 9\sigma\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\right.$$

$$\left. + \frac{2}{\eta_1}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}} - \frac{2\eta_2}{\eta_1^3}\left(\frac{1+3\sigma\eta_1}{\gamma_1\eta_1}\right)^{\frac{1}{2}}\right\}.$$

Then, the coefficients P and Q are given by

$$P = \frac{-k^2}{2\widetilde{\omega}\gamma_1(k^2+\eta_1)^3}\left\{-(k^2-3\eta_1) + \frac{(k^2+\eta_1)^3}{k^2}\left(\frac{\widetilde{\omega}^2\gamma_1}{k^2} - \frac{2\widetilde{\omega}\gamma_1\tilde{\lambda}}{k} + \gamma_1\tilde{\lambda}^2\right)\right\}$$

$$\approx -\frac{1}{2}\frac{k}{\gamma_1\eta_1^3}\left[\frac{(3\sigma\eta_1+1)}{\gamma_1\eta_1}\right]^{\frac{-1}{2}}(3\eta_1) = -\frac{3}{2}\frac{k}{\eta_1^2}\left[\frac{\gamma_1(3\sigma\eta_1+1)}{\eta_1}\right]^{\frac{-1}{2}} = -\frac{3}{2}\frac{b}{\eta_1^2}k$$

$$Q = \frac{-k^2}{2\widetilde{\omega}\gamma_1}\left\{\left(\frac{\widetilde{\omega}^2\gamma_1}{k^2} + 3\sigma\right)(A_n + B_n) + \frac{2\widetilde{\omega}}{k}\left(\gamma_1 - \frac{\widetilde{\omega}}{k}\gamma_2\right)(A_u + B_u) - \frac{2\eta_2}{\left(k^2 + \eta_1\right)^2}(A_\Phi + B_\Phi)\right.$$
$$\left. + 2\left(\frac{\widetilde{\omega}}{k}\right)^3\gamma_2 - \frac{3}{2C^2}\frac{\widetilde{\omega}}{k}\left(\frac{\widetilde{\omega}}{k}\right)^3 - \frac{3\eta_3}{\left(k^2 + \eta_1\right)^4}\right\}$$
$$\approx \frac{-k}{2\gamma_1}\left(\frac{3\sigma\eta_1 + 1}{\gamma_1\eta_1}\right)^{\frac{-1}{2}}\left\{\left(\frac{\widetilde{\omega}^2\gamma_1}{k^2} + 3\sigma\right)(A_n + B_n) + \frac{2\widetilde{\omega}}{k}\left(\gamma_1 - \frac{\widetilde{\omega}}{k}\gamma_2\right)(A_u + B_u)\right.$$
$$\left. - \frac{2\eta_2}{\eta_1^2}(A_\Phi + B_\Phi)\right\}$$
$$\approx \frac{\eta_1^2}{3k}\left[\frac{\gamma_1\left(3\sigma\eta_1 + 1\right)}{\eta_1}\right]^{\frac{-1}{2}}\left\{\frac{\left(3\sigma\eta_1 + 1\right)\left(3\eta_1^2 - 2\eta_2\right)}{2\eta_1^3} + \frac{3}{2}\frac{\sigma\left(2\eta_2 + \eta_1^2\right)}{\eta_1^2}\right.$$
$$\left. - \frac{\gamma_2}{\gamma_1^{\frac{3}{2}}}\left(\frac{3\sigma\eta_1 + 1}{\eta_1}\right)^{\frac{3}{2}}\right\}^2 = \frac{1}{3k}\frac{a^2\eta_1^2}{b}$$

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