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PROPERTIES AND APPLICATIONS OF THE COMPOSITION OPERATOR BETWEEN CERTAIN FUNCTION SPACES

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ABSTRACT

In this study, we aim to investigate the boundedness and compactness of the composition operator C_Θ, and to provide characterizations of its boundedness and compactness. We introduce necessary conditions and sufficient conditions for the operator C_Θ to map $\mathcal{B}_g^{(m,n)}$ into the Q_P space. Composition operator C_Θ , the Space Q_P , boundedness and compactness, the space $\mathcal{B}_g^{(m,n)}$.

Key Words:

Composition operator C_Θ, the Space Q_P , boundedness and compactness, the space $\mathcal{B}_g^{(m,n)}$.

1. INTRODUCTION

Numerous investigations of analytic function spaces by several different types of functions are introduced and subjected to in-depth investigation. Many active subfields of mathematics, particularly those of mathematical analysis, such as operator theory, measure theory, and differential equations, can use some of the intriguing tools use of some of the intriguing tools that are made available by the theory of function spaces.

This operator maps from a space denoted as $\mathcal{B}_g^{(m,n)}$ to another space called Q_p spaces. The study aims to explore and identify the characteristics that make this operator bounded or compact. In particular, the research focuses on developing equivalent characterizations for the properties of the C_Θ operator. This will involve analyzing the properties of both the source and target spaces and studying the behavior of the operator under different conditions. By doing so, the study aims to provide a more complete understanding of the composition operator and its applicability in various mathematical contexts.

Additionally, In complex analysis, the unit disc in the complex plane is a set of complex numbers defined as the set of all complex numbers with an absolute value less than 1. It is denoted by $\mathbb{D} = \{\mathcal{G} \in \mathbb{C}: |\mathcal{G}| < 1\}$. The unit disc is an example of an open set in the complex plane and is often used as a domain for studying complex analysis. The boundary of the unit disc is the unit circle, which is the set of all complex numbers with an absolute value of 1, i.e., $|\mathcal{G}| = 1$. The unit disc plays an important role in complex analysis as it is a simply connected domain, meaning that any two paths with the same endpoints can be continuously deformed into each other without leaving the domain.

The definition of the Möbius transformations function denoted by $\varphi_\vartheta(\mathcal{G})$ can be obtained from [1] as follows:

for each $\mathcal{G} \in \mathbb{D}$, $\varphi_\vartheta: \mathbb{D} \to \mathbb{D}$

$$\varphi_\vartheta(\mathcal{G}) := \frac{\vartheta - \mathcal{G}}{1 - \bar{\vartheta}\mathcal{G}}, \qquad for \quad every \quad \mathcal{G} \in \mathbb{D}.$$

Möbius transformations are one-to-one and onto transformations of the extended complex plane, which includes the point at infinity. They can be used to map circles and lines to other circles and lines, and to transform more general shapes in the complex plane. Möbius transformations have many applications in complex analysis, geometry, and physics, and they are also used extensively in computer graphics and computer vision. The following identities are easily verified.

$$1 - |\varphi_\vartheta(\mathcal{G})|^2 = \frac{(1 - |\vartheta|^2)(1 - |\mathcal{G}|^2)}{|1 - \bar{\vartheta}\mathcal{G}|^2},$$

$$|\varphi'_\vartheta(\mathcal{G})| = \frac{(1 - |\varphi_\vartheta(\mathcal{G})|^2)}{(1 - |\mathcal{G}|^2)} = \frac{(1 - |\vartheta|^2)}{|1 - \bar{\vartheta}\mathcal{G}|^2},$$

and

$$\frac{1 - |\varphi_a(\mathcal{G})|^2}{1 - |\mathcal{G}|^2} \leq \frac{4}{1 - |\vartheta|^2}.$$

The Green's function of the open unit disk $\mathbb{D}$ with a logarithmic singularity at a point ϑ is defined as follows in complex analysis:

$$g(\mathcal{G}, \vartheta) := \log\left|\frac{1 - \bar{\vartheta}\mathcal{G}}{\mathcal{G} - \vartheta}\right| = \log\frac{1}{|\varphi_\vartheta(\mathcal{G})|}.$$

For each $\vartheta \in \mathbb{D}$, the pseudo-hyperbolic disc $D(\vartheta, r)$ is defined as follows:

$$D(\vartheta, r) = G \in \mathbb{D}: |\varphi_\vartheta(G)| < r,$$

where $\varphi_\vartheta(G) = \frac{\vartheta - G}{1 - \bar{\vartheta}G}$ is the M"obius transformation that maps $\mathbb{D}$ onto itself with $\varphi_\vartheta(\vartheta) = 0$ and $\varphi_\vartheta(0) = \vartheta$. The parameter r is a positive number such that $0 < r < 1$. The Green's function of $\mathbb{D}$ with a logarithmic singularity is useful in the study of harmonic functions and conformal mappings in the unit disc. The pseudo-hyperbolic disc $D(a, r)$ plays an important role in the study of Besov spaces and other function spaces on the unit disc (see [2]).

The operator C_Θ has been researched in various Banach spaces of analytic functions(see[3, 4, 5, 6, 7] and others), If a reader is interested in this subject, he can find more information at (see[8, 9]) references to original works that laid the groundwork for the advancement of the theory related to composition operators and function spaces.

The linear composition operator C_Θ can be defined as follow: $C_\Theta(\mho) := \mho \circ \Theta$ (see[10]).

The analytic Bloch-type space $\mathcal{B}$ is a subset of the Bloch space, and is defined as follows:

$$\mathcal{B} = \{\mho \in H(\mathbb{D}): \sup_{\mathcal{G} \in \mathbb{D}} (1 - |\mathcal{G}|^2)|\mho'(\mathcal{G})| < \infty\}.$$

where $H(\mathbb{D})$ denotes the set of all analytic functions on the open unit disk $\mathbb{D}$, and $\mho'(\mathcal{G})$ denotes the derivative of the function $\mho$ at the point $\mathcal{G}$. The analytic Bloch-type space $\mathcal{B}$ has many interesting properties and applications in complex analysis and operator theory. For example, it has been used to study the boundedness and compactness of composition operators on various function spaces, and to characterize the boundedness of Hankel operators on weighted Bergman spaces. It has also been used to study the boundary behavior of functions in $\mathcal{B}$, and to prove various types of approximation theorems see [11, 12, 13, 14, 2, 15].

In addition to the analytic Bloch-type space $\mathcal{B}$, we can define the little Bloch-type space $\mathcal{B}_0$ as follows:

$$\mathcal{B}_0 = \{\mho \in H(\mathbb{D}): \lim_{|\vartheta|\to 1^-} (1-|\mathcal{G}|^2)|\mho'(\mathcal{G})| = 0\}.$$

We refer to [16, 17, 18, 19, 20, 21] and other sources for the various global research conducted on Bloch-type spaces.

The analytic Q_p spaces are a family of function spaces on the unit disk $\mathbb{D}$, defined in terms of a weight function $g(\mathcal{G}, a)$ that depends on a fixed point $a \in \mathbb{D}$ and a parameter $p > 0$. Specifically, The Q_p space was first introduced in [11] as follows:

Definition 1 *Let $0 < p < \infty$ and let G be an analytic function on $\mathbb{D}$. Then the analytic Q_p space is defined as*

$$Q_p := \{\mho \in H(\mathbb{D}): \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} |\mho'(\mathcal{G})|^2 g^p(\mathcal{G}, a) dA(\mathcal{G}) < \infty\},$$

where $dA(\mathcal{G})$ is the normalized area measure on $\mathbb{D}$ and $g(\mathcal{G}, a)$ is the weight function defined by

$$g(\mathcal{G}, a) = \frac{1}{|1-\overline{a}\mathcal{G}|^2}.$$

Equivalently, we can define Q_p using the weight function

$$g(\mathcal{G}, a) = (1-|\Theta_a(\mathcal{G})|)^p,$$

where $\Theta_a(\mathcal{G})$ is the automorphism of $\mathbb{D}$ that maps a to 0.

The norm of Q_p can be defined as follows:

$$||\mho||_{Q_p} = |\mho(0)| + Q_p(\mho)$$

The Q_p spaces are important in several areas of complex analysis, including the theory of Bergman spaces, Hardy spaces, and Bloch spaces. They have been extensively studied in the context of function theory on the unit disk, and have been used to study the boundedness and compactness of various operators on function spaces.

For a more comprehensive study of analytic Q_p spaces, one can refer to [11, 22, 23, 24, 25] and other related works. Furthermore, it is possible to extend these weighted classes of analytic functions in C^n, as demonstrated in works by Essen and Feng [26, 27].

On the other hand, there are some interesting extensions using quaternion-valued functions setting Hereafter, we set

$$\varphi_{\mathcal{G}_0}(\mathcal{G}) := \frac{\mathcal{G}_0 - \mathcal{G}}{1-\bar{\mathcal{G}}_0\mathcal{G}}, \qquad for \quad every \quad \mathcal{G} \neq \mathcal{G}_0.$$

and set

$$\varphi_{\mathcal{G}_0}(\mathcal{G}) = C < 1, \; when \; \mathcal{G} = \mathcal{G}_0.$$

The function of the modified Green's function is defined as follow:

$$g(\mathcal{G}, \mathcal{G}_0) := \ln\left|\frac{1-\bar{\mathcal{G}}_0\mathcal{G}}{\mathcal{G}_0-\mathcal{G}}\right| = \ln\frac{1}{|\varphi_{\mathcal{G}_0}(\mathcal{G})|}.$$

Following are some definitions that can be offered, all of which are motivated by the modified Green's function.

The analytic g-Bloch space $\mathcal{B}_g^{(m,n)}$ is a space of holomorphic functions defined on the unit disk $\mathbb{D}$ that satisfy a certain growth condition with respect to the function g and its derivatives. Specifically, for given constants $0 < m < \infty$, $0 < n < \infty$, and $0 < \alpha < \infty$, a function $\mho \in H(\mathbb{D})$ belongs to $\mathcal{B}_g^{(m,n)}$ if there exists a constant $C > 0$ such that

$$\sup_{\mathcal{G},\mathcal{G}_0 \in \mathbb{D}} \frac{(1-|\mathcal{G}|^2)^n}{g^m(\mathcal{G},\mathcal{G}_0)} |\mho'(\mathcal{G})| < C.$$

In other words, the derivative of $\mho$ satisfies a certain growth condition with respect to the function g and its derivatives.

The analytic g-Bloch space $\mathcal{B}_g^{(m,n)}$ is a natural generalization of the classical Bloch space and other related spaces, such as the little Bloch space. The properties and applications of these spaces have been extensively studied in the literature, and they have important connections to various areas of analysis, including complex analysis, harmonic analysis, and operator theory.

Definition 2 *see[28] For given constants* $0 < m < \infty$, $0 < n < \infty$, *and* $0 < \alpha < \infty$, *we can define the analytic* g*-Bloch space* $\mathcal{B}_g^{(m,n)}$ *for a function* ϑ *as follows:*

$$\mathcal{B}_g^{(m,n)} = \{\mho \in H(\mathbb{D}): \sup_{\mathcal{G},\mathcal{G}_0 \in \mathbb{D}} \frac{(1-|\mathcal{G}|^2)^n}{g^m(\mathcal{G},\mathcal{G}_0)} |\mho'(\mathcal{G})| < \infty\}.$$

The space $\mathcal{B}_g^{(m,n)}$ with the norm is defined as follows:

$$||\mho||_{\mathcal{B}_g^{(m,n)}} = |\mho(0)| + \mathcal{B}_g^{(m,n)}(\mho).$$

2. Auxiliary Results

Site Description: In this section, we state several results, which are used in the main result proofs. Now, we will introduce the definition of boundedness and compactness of the operator $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$.

Definition 3 *The operator* $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$ *is considered to be bounded if there exists a positive constant* C *such that for all* $\mho \in \mathcal{B}_g^{(m,n)}$, *we have* $||C_\Theta \mho|| Q_p \leq C ||\mho||_{\mathcal{B}_g^{(m,n)}}$.

Definition 4 *The operator* $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$ *is said to be compact if the image of any unit disc in* $\mathcal{B}_g^{(m,n)}$ *under* C_Θ *is a pre-compact set in* Q_p.

We present a list of the following lemmas, all of which are required to demonstrate our primary results.

Lemma 5 *if* $\mho_1, \mho_2 \in \mathcal{B}_g^{(m,n)}$ *and* $n, m \in \mathbb{N}$ *satisfy the inequality*

$$[|\mho'_1(\mathcal{G})| + |\mho'_2(\mathcal{G})|] \geqslant \frac{C g^m(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^n},$$

however, the proof follows a similar approach to Lemma 1 in [29], and thus, we have omitted it in this context.

3. THE BOUNDEDNESS OF THE OPERATOR $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$

In the following section, we characterize the operators $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$. In addition, we provide the conditions that prove the boundedness of the operators C_Θ. Now we'll go over the main boundedness results.

Theorem 6 Let g is an analytic function in $\mathbb{D}$ and Θ is an analytic self-map of $\mathbb{D}$, then $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$ is bounded if and only if

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\Theta\prime(\mathcal{G})|^2 g^{p+2m}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n}} dA(\mathcal{G}) < \infty. \tag{1}$$

Proof. For $\mho \in \mathcal{B}_g^{(m,n)}$ with $\| \mho \|_{\mathcal{B}_g^{(m,n)}} \leq 1$ then, we obtain

$$\begin{aligned}
&\| C_\Theta \mho(\mathcal{G}) \|_{Q_p} = \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} |(\mho \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&= \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} |(\mho'(\Theta(\mathcal{G}))\Theta'(\mathcal{G}))|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&\leq \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} |\mho'(\Theta(\mathcal{G}))|^2 |\Theta'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&\leq \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\mho\prime(\Theta(\mathcal{G}))|^2 |\Theta\prime(\mathcal{G})|^2 (1-|\mathcal{G}|^2)^{2n} g^{p+2m}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n} g^{2m}(\mathcal{G},\mathcal{G}_0)} dA(\mathcal{G}) \\
&\leq \| \mho \|_{\mathcal{B}_g^{(m,n)}} \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\Theta\prime(\mathcal{G})|^2 g^{p+2m}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n}} dA(\mathcal{G}) \\
&< \infty.
\end{aligned} \tag{2}$$

To prove the converse direction, we utilize the fact that the function $C_\Theta \mho(\mathcal{G})$ belongs to Q_p for any $\mho \in \mathcal{B}_g^{(m,n)}$, and then apply Lemma 5 to obtain the following:

$$\begin{aligned}
&4\{\| C_\Theta \mho_1(\mathcal{G}) \|_{Q_p} + \| C_\Theta \mho_2(\mathcal{G}) \|_{Q_p}\} \\
&= 4\sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} [|(\mho_1 \circ \Theta)'(\mathcal{G})|^2 + |(\mho_2 \circ \Theta)'(\mathcal{G})|^2] g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&= 4\sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} [|(\mho_{1\prime}(\Theta(\mathcal{G}))\Theta'(\mathcal{G}))|^2 + |(\mho_{2\prime}(\Theta(\mathcal{G}))\Theta'(\mathcal{G}))|^2] g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&= \sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} [|\mho_{1\prime}(\Theta(\mathcal{G}))|^2 + |\mho_{2\prime}(\Theta(\mathcal{G}))|^2] |\Theta'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&\geqslant 4\sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} \frac{[|\mho_{1\prime}(\Theta(\mathcal{G}))| + |\mho_{2\prime}(\Theta(\mathcal{G}))|]^2 |\Theta\prime(\mathcal{G})|^2 (1-|\mathcal{G}|^2)^{2n} g^{p+2m}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n} g^{2m}(\mathcal{G},\mathcal{G}_0)} dA(\mathcal{G}) \\
&\geqslant C\sup_{\mathcal{G}\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\Theta\prime(\mathcal{G})|^2 g^{p+2m}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n}} dA(\mathcal{G}).
\end{aligned} \tag{3}$$

Hence the operator C_Θ is bounded, Hence (1) holds. We have finished our proof.

The composition operator $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$ is compact if and only if for every sequence $\{\mho_n\}_{n\in\mathbb{N}} \subset Q_p$ bounded in Q_p norm and $\mho_n \to 0, n \to \infty$, uniformly on compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence

$$\| C_\Theta(\mho_n) \|_{Q_p} \to 0, n \to \infty.$$

The following result describes compactness.

4. THE COMPACTNESS OF THE OPERATOR $C_\Theta: \mathcal{B}_g^{(m,n)} \to Q_p$

Lemma 7 Let Θ be an analytic self-map of the unit disk $\mathbb{D}$, and let h belong to the space of holomorphic functions on $\mathbb{D}$. Then, the composition operator $C_\Theta: \mathcal{B}^{(m,n)}g \to Q_p$ is said to be compact if and only if it is bounded and for any bounded sequence$\{\mho_i\}_{i\in N} \in \mathbb{N} \subset \mathcal{B}^{(m,n)}g$ that converges uniformly to zero on compact subsets of $\mathbb{D}$, we have $\lim_{i\to\infty} |C_\Theta \mho_i|_{Q_p} = 0$.

Theorem 8 *The operator $C_\Theta : \mathcal{B}_g^{(m,n)} \to Q_p$ is a compact operator if and only if $\Theta \in Q_p$, where Θ is an analytic self-map of the unit disk and*

$$\lim_{r\to 1}\sup_{\mathcal{G}\in\mathbb{D}} \frac{|\Theta\prime(\mathcal{G})|^2 g^{(2m+p)}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n}} dA(\mathcal{G}) = 0. \quad (4)$$

Proof. Assume that $C_\Theta : \mathcal{B}_g^{(m,n)} \to Q_p$ is a compact operator, where $\Theta \in Q_p$. Let U_r^1 and U_r^2 be subsets of $\mathbb{D}$ defined by $|\Theta(\mathcal{G})| > r$ and $|\Theta(\mathcal{G})| \leqslant r$ for $r \in (0,1)$, respectively. Define $\mho_n(\mathcal{G}) = \frac{\mathcal{G}^n}{n}$. Since $\| \mho \| \mathcal{B}_g^{(m,n)} \leqslant M$ and $\mho_n(\mathcal{G}) \to 0$ as $n \to \infty$ locally uniformly on the unit disk, then $\| C_\Theta(\mho_n) \|_{Q_p} \to 0, n \to \infty$.

For any $r \in (0,1)$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$\frac{N^p}{r^{p(1-N)}} \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\Theta'(\mathcal{G})|^2 (1-|\mathcal{G}|^2)^{2n} g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < \epsilon.$$

Furthermore, if we choose r such that $\frac{N^p}{r^{p(1-N)}} = 1$, then

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\Theta'(\mathcal{G})|^2 (1-|\mathcal{G}|^2)^{2n} g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < \epsilon. \quad (5)$$

Assuming that $\mho$ satisfies $\| \mho \|_{\mathcal{B}_g^{(m,n)}} \leqslant 1$, we define a family of functions $\mho_t(\mathcal{G}) = \mho(t\mathcal{G}), \quad t \in (0,1)$. It follows that $\mho_t \to \mho$ uniformly on compact subsets of the unit disk as $t \to 1$, and $(\mho_t)$ is bounded on $\mathcal{B}_g^{(m,n)}$. We then have

$$\| ((\mho_t \circ \Theta)'(\mathcal{G}))^2 - ((\mho \circ \Theta)'(\mathcal{G}))^2 \| \to 0$$

as $t \to 1$. Because C_Θ is compact, we can choose $\epsilon > 0$ such that

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\mho_t(\Theta(\mathcal{G}))|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < \epsilon,$$

where

$$(\mho_t(\Theta(\mathcal{G})))^2 = ((\mho \circ \Theta)'(\mathcal{G}))^2 - ((\mho_t \circ \Theta)'(\mathcal{G}))^2.$$

Therefore, when we fix t, then

$$\begin{aligned}
&\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&= \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho'(\Theta(\mathcal{G}))\Theta'(\mathcal{G}))|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&= \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\mho'(\Theta(\mathcal{G}))|^2 |\Theta'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&\leq 4\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\mho_t(\Theta(\mathcal{G}))|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&+ 4\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho_t \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&= 4\epsilon + 4\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\mho_{t'}(\Theta(\mathcal{G}))|^2 |\Theta'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&\leq 4\epsilon + 4 \| \mho'_t \|_\infty^2 \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\Theta'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\
&\leq 4\epsilon + 4 \| \mho'_t \|_\infty^2, \quad (6)
\end{aligned}$$

i.e.,

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \leq \epsilon + 2 \| \mho'_t \|_\infty^2. \quad (7)$$

with respect to (5), there exists a δ dependent on $\mho$ and ϵ for each $\| \mho \|_{B_g^{(m,n)}}$ and $\epsilon > 0$, such that for r belonging to the interval $[\delta, 1)$, where we have used (5). On the other hand, for each $\| \mho \|_{B_g^{(m,n)}}$

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < \epsilon. \tag{8}$$

Due to the compactness of the operator C_Θ, it maps the unit disk of $B_g^{(m,n)}$ to a subset of Q_p that is relatively compact. Consequently, for any $\epsilon > 0$, there exists a finite set of functions $\mho_1, \mho_2, \dots, \mho_n$ in the unit disk of $B_g^{(m,n)}$ such that, for each $\| \mho \|_{B_g^{(m,n)}} \leq 1$, there exists $k \in 1,2,3,\dots,n$ where

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |\mho_K(\Theta(\mathcal{G}))|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < \epsilon,$$

and

$$(\mho_k(\Theta(\mathcal{G})))^2 = ((\mho \circ \Theta)'(\mathcal{G}))^2 - ((\mho_k \circ \Theta)'(\mathcal{G}))^2.$$

Also, by using (8), we get for $\delta = max_{1\leq k\leq n}\delta(\mho_k, \epsilon)$ and $r \in [\delta, 1)$, that

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho_k \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < \epsilon.$$

Hence for any $\mho, \| \mho \|_{B_g^{(m,n)}} \leq 1$, the two relations can be combined to get

$$\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) < 2\epsilon.$$

As a result, we can conclude that (4) is valid. To establish sufficiency, we utilize the fact that $\Theta \in Q_p$ and (4) holds. Suppose we have a sequence of functions $\mho_{n_{n\in\mathbb{N}}}$ in the unit disk of $B_g^{(m,n)}$ such that $\mho_n \to 0$ uniformly on the compact subsets of the unit disk as $n \to \infty$. Let $r \in (0,1)$. Then,

$$\begin{aligned} &\| C_\Theta \mho_n(\mathcal{G}) \|_{Q_p} \leq 2|\mho_n(\Theta(0))| + 2\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho_n \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\ &+2\sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^2} |(\mho_n \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\ &= 2(I_1 + I_2 + I_3). \end{aligned} \tag{9}$$

As $\mho_n$ tends to zero locally uniformly on the unit disk as n approaches infinity, we have $I_1 = |\mho_n(\Theta(0))|$ approaching zero as n tends to infinity. Furthermore, for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\begin{aligned} I_2 &= \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^2} |(\mho_n \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\ &\leq \epsilon \| \Theta \|_{Q_p}. \end{aligned} \tag{10}$$

We also observe that

$$\begin{aligned} I_3 &= \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} |(\mho_n \circ \Theta)'(\mathcal{G})|^2 g^p(\mathcal{G},\mathcal{G}_0) dA(\mathcal{G}) \\ &\leq \| \mho_n \|_{B_g^{(m,n)}} \sup_{\mathcal{G}\in\mathbb{D}} \int_{U_r^1} \frac{|\Theta'(\mathcal{G})|^2 g^{(2m+p)}(\mathcal{G},\mathcal{G}_0)}{(1-|\mathcal{G}|^2)^{2n}} dA(\mathcal{G}). \end{aligned} \tag{11}$$

By using that (4) holds, then for any $n > N$ and $\epsilon > 0$, there exists r_1 such that $I_3 < \epsilon$ for all $r > r_1$. Consequently, if Θ belongs to Q_p, then we have the following:

$$\begin{aligned} &\| C_\Theta \mho_n(\mathcal{G}) \|_{Q_p} \leq 2(0 + \epsilon \| C_\Theta \|_{Q_p} + \epsilon) \\ &\leq C\epsilon. \end{aligned} \tag{12}$$

By combining the previously mentioned results, we can conclude that

$$\| C_\Theta(\mho_n) \| Q_p \to 0, n \to \infty.$$

This establishes the compactness theorem for the composition operator C_Θ.

5. REAL APPLICATIONS

The study of this composition operator's properties, such as boundedness and compactness, can lead to applications in several fields see[30, 31, 32, 33].

Complex Dynamics: The study of the behavior of holomorphic self-maps of the open unit disk and their orbits can provide insights into the dynamics of the composition operator. This can help understand how different functional spaces interact with each other under the action of a composition operator.

Control Theory: Composition operators can be used to model the behavior of dynamical systems and provide insights into the system's stability and control. Understanding the properties of composition operators between different functional spaces can be crucial in designing control systems and analyzing their performance.

Signal Processing: The study of composition operators between g-Bloch space and Q_p can help develop novel techniques for signal processing and analysis. For example, these operators can be used to model the behavior of filters in signal processing applications or design wavelets with specific properties.

Function Theory: The interaction between g-Bloch and Q_p spaces via composition operators can lead to a deeper understanding of function spaces and their properties. It may result in the discovery of new functional spaces, inequalities, or embeddings with potential applications in other areas of mathematics.

Operator Theory: Analyzing the properties of composition operators between g-Bloch and Q_p spaces can also help develop new results in operator theory. For example, this can lead to a better understanding of other operator classes (e.g., Toeplitz operators, Hankel operators) and their connections to functional spaces.

While these applications may not have direct, everyday implications, they contribute to the development of the broader mathematical framework and can eventually impact various fields, such as engineering, physics, and computer science.

6. CONCLUSION

The focus of this study is to explore the boundedness and compactness of the composition operator that maps from $\mathcal{B}_g^{(m,n)}$ space to Q_p spaces defined on the unit disk. In addition to investigating these properties, we have also derived necessary and sufficient conditions for the mapping from $\mathcal{B}_g^{(m,n)}$ to Q_P spaces.

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