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Fully nonlinear dust-ion-acoustic excitation in Jupiter ionosphere

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ABSTRACT

Fully nonlinear solitary wave structure of dust-ion-acoustic waves are investigated in a five-component plasma consisting of positive proton beam from solar wind, positive ion fluid, two electrons population, one of them from the solar wind and stationary positive dust grains. The physical parameters in the system such as, proton beam-to-positive ion temperature and density ratios, as well as solar wind electron number density play an important role in the profile of the large amplitude dust-ion-acoustic solitary waves. Using pseudo-potential approach (Sagdeev potential) the basic equations are reduced to one evolution equation called an energy equation. The latter has been analyzed and solved numerically to obtain an arbitrary amplitude solitary profile as well as the possible regions for the existence waves. The solution of energy equation presents a positive potential, which corresponds to a compressive wave profile. The findings of this investigation are used to interpret the electrostatic arbitrary solitary waves that may be observed in the Jupiter ionosphere. dust-ion-acoustic waves; Jupiter ionosphere; large amplitude solitary waves

Key Words:

dust-ion-acoustic waves; Jupiter ionosphere; large amplitude solitary waves.

1. INTRODUCTION

The propagation of slow frequency wave in classical plasma, which known as ion-acoustic wave (IAW) in astrophysical and laboratory plasma was creating a lot of attentions. Tonks and Langumir firstly foretell the existence of the IAWs [1], while ravines turn out the first experiment to observe these waves [2]. In 1970, these waves are observed in laboratory by Ikezi et al [3]. Viking and Frejo spacecrafts observed and described the solitary structure in the magnetosphere [4]. In recent years, most of studies of the nonlinear IAWs lead to different kinds of nonlinear waves, such as solitary waves, double layers, super-solitons, shocks, and rouge waves. The propagation of IAWs was studied in a magnetized three component plasma by Mahmood et al. [5], where they studied the propagation speed of soliton transformed to be subsonic waves. In an unmagnetized Argon plasma, Suryanarayana et al. [6] studied theoretically the propagation of IAWs by using a small amplitude technique and experimental setting. Further, investigations in nonlinear and linear IAWs in plasma and dusty plasma are continued in many works [7, 8, 9, 10, 11, 12, 13]. The Sagdeev pseudo-potential is one of the pioneer processes in studying arbitrary amplitude waves in either plasma or dusty plasma. Ghosh et al [14] used Sagdeev pseudo-potential to study two-component plasma with cold electrons and warm ions. The investigation of Sagdeev electrostatic potential in two component plasmas consisting of trapped electrons and negative ions is studied by Roychoudhury et. al. [15]. Mamun [16] used this method to study multi-components dusty plasma composed Maxwellian ions and electrons as well as charged dust fluid. Two-component dusty plasma with non-thermally distributed ions and negatively charged dust fluid was studied by Mendoza-Bricenoa et al. [17] and the coexistence of compressive and rarefactive solitons was due to the non-thermal ions. Baluku et al [18] investigated the observed large amplitude dust-acoustic solitary waves (DAWs) in a three-component dusty plasma with kappa distributed ions, electrons and cold negative dust grains. The study of arbitrary amplitude IAWs using the Sagdeev potential method was continued by [19, 20, 21, 22, 23, 24].

Understanding the fully nonlinear solitary waves in Jupiter magnetosphere is important for space physics investigations. This non-linearity depends on the properties of all the physical plasma parameters as well as on the nature of the planetary obstacle. Investigations of the plasma phenomena in the Jupiter ionosphere using fluid equations and magnetohydrodynamic equations are of interest in this work. Therefore, the objective of this paper is to study the existence and propagation of large amplitude dust-ion-acoustic solitary waves in an unmagnetized collisionless plasma system composed of five different population charged particles. The manuscript is organized as follow. Section II, the basic set of fluid equations that describes the Jupiter ionosphere is presented. Section III, The derivation of Sagdeev pseudo-potential is obtained for arbitrary amplitude waves. The effect of physical parameters on the existence region and solitary wave profile are presented in Section IV. Finally a summary of the present work is written in Section V.

2. Basic equations:

We consider five components collisionless, unmagnetized plasma in Jupiter ionosphere environment having two positive ions (one kind of ions from Jupiter (i) and the other from the solar wind (p)), two Maxwellian electrons (one kind of electrons from Jupiter (e)and the other from the solar wind (es)), and stationary positive dust impurities in Jupiter [25]. The normalized continuity and momentum equations for ions are describe as

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{\partial \varphi}{\partial x} + \sigma_1 n_i \frac{\partial n_i}{\partial x} = 0, \tag{2}$$

$$\frac{\partial n_p}{\partial t} + \frac{\partial}{\partial x}(n_p u_p) = 0, \tag{3}$$

$$\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial x} + \frac{\sigma_2}{Q} n_p \frac{\partial n_p}{\partial x} + \frac{1}{Q} \frac{\partial \varphi}{\partial x} = 0. \quad (4)$$

The pressure gradient force are considered as adiabatic pressure. The Maxwellian electrons in the Jupiter and solar wind are expressed, respectively, as

$$n_e = \exp(\varphi), \quad (5)$$

$$n_{es} = \exp(\sigma_{es}\varphi). \quad (6)$$

Equations (1)–(6) are closed by the Poisson equation

$$\frac{\partial^2 \varphi}{\partial x^2} + n_i + \mu_p n_p - \mu_e n_e - \mu_{es} n_{es} + \alpha = 0. \quad (7)$$

where $\mu_p = \frac{n_{p0}}{n_{i0}}$, $\mu_e = \frac{n_{e0}}{n_{i0}}$, $\mu_{es} = \frac{n_{es0}}{n_{i0}}$, $\alpha = \frac{Z_{d0} n_{d0}}{n_{i0}}$ are the ratios of unperturbed charges densities-to-ion density. u_i; and u_p are the fluid velocities of the positive ion and solar wind proton beam, respectively, which are normalized by the ion-acoustic speed $C_s = \sqrt{K_B T_e/m_i}$, φ is the electrostatic potential is normalized by $(k_B T_e/e)$. The space coordinate x and the time t are normalized by the Debye length $\lambda_{Di} = (k_B T_e/(4\pi n_{i0} e^2))^{1/2}$ and the inverse of ion plasma frequency $\omega_{pi} = (m_1/(4\pi n_{i0} e^2))^{-1/2}$ $\sigma_1 = (3T_i/T_e)$, $\sigma_2 = (3T_p/T_e)$, $\sigma_{es} = (T_{es}/T_e)$, $Q = m_p/m_i$. We are interested to consider the ion wave motion. At this scale the dust are too heavy to fellow the moving ions and it considers as stationary with respect to ions.

The charge neutrality condition in the plasma is always maintained through the relation

$$n_{p0} + n_{i0} - n_{e0} - n_{es0} + Z_{d0} n_{d0} = 0, \quad (8)$$

where n_{p0}, n_{i0}, n_{e0}, n_{es0}, and n_{d0} are the initial/unperturbed densities of the charged particles.

3. Derivation of evolution equation

To study the characteristics of large amplitude dust-ion-acoustic waves in the upper Jupiter ionosphere, we introduce the following traveling wave transformation $\xi = x - Mt$, where M is the Mach number normalized to the ion-acoustic speed C_s. Substituting this transformation into the set of equations (1)–(7) we have

$$-M \frac{dn_i}{d\xi} + \frac{d}{d\xi}(n_i u_i) = 0, \quad (9)$$

$$-M \frac{du_i}{d\xi} + u_i \frac{du_i}{d\xi} + \frac{d\varphi}{d\xi} + \sigma_1 n_i \frac{dn_i}{d\xi} = 0, \quad (10)$$

$$-M\frac{dn_p}{d\xi}+\frac{d}{d\xi}\left(n_p u_p\right)=0, \tag{11}$$

$$-M\frac{du_p}{d\xi}+u_p\frac{du_p}{d\xi}+\frac{\sigma_2}{Q}\mu n_p\frac{dn_p}{d\xi}+\frac{1}{Q}\frac{d\varphi}{d\xi}=0, \tag{12}$$

$$\frac{d^2\varphi}{d\xi^2}=\mu_e n_e+\mu_{es}n_{es}-\mu_p-n_i-\alpha. \tag{13}$$

These equations satisfy the boundary conditions

$$\xi\to\infty, n_i\to 1, n_p\to 1, u_i\longrightarrow 0,\ u_p\to u_{p0}, \varphi\to 0, \text{and}\frac{d\varphi}{d\xi}\to 0, \tag{14}$$

Integrating equations (9) and (10) using these boundary conditions, we get

$$n_i=\frac{1}{6\sqrt{3\sigma_1}}\begin{pmatrix}\left(\left(M-\sqrt{3\sigma_1}\right)^2-2\varphi\right)^{3/2}-\left(\left(M+\sqrt{3\sigma_1}\right)^2-2\varphi\right)^{3/2}-\\ \left(M-\sqrt{3\sigma_1}\right)^3-\left(M+\sqrt{3\sigma_1}\right)^3\end{pmatrix}. \tag{15}$$

The sign inside the root is taken negative to satisfy the boundary condition for the number density of ions which is unity. Integrating equations (11) and (12) using these boundary conditions, we have

$$n_p=\frac{1}{6\sqrt{3\sigma_2 Q}}\begin{pmatrix}\left(\left(M-u_{p0}+\sqrt{3\sigma_2/Q}\right)^2-2\varphi/Q\right)^{3/2}-\left(\left(-M+u_{p0}+\sqrt{3\sigma_2/Q}\right)^2-2\varphi/Q\right)^{3/2}\\ -\left(\left(M-u_{p0}+\sqrt{3\sigma_2/Q}\right)^2-\left(-M+u_{p0}+\sqrt{3\sigma_2/Q}\right)^2\right)\end{pmatrix}, \tag{16}$$

Substituting from equations (15), (5), and (6) into equation (9)–(13), and integrating, gives the Hamiltonian function for this system as

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2+S(\varphi)=0. \tag{17}$$

This equation represents the energy equation of an oscillating particle of unit mass and has velocity $d\varphi/d\xi$ at position φ moving under the effect of an arbitrary amplitude Sagdeev pseudo-potential (AASPP) $S(\varphi)$, which is given by

$$S(\varphi)=n_i+\mu_p n_p+A_1, \tag{18}$$

where

$$A_1 = -\mu_e(\exp(\varphi) - 1) - \mu_{es}\big(-1 + \exp(\sigma_{es}\varphi)\big) + \alpha\varphi. \quad (19)$$

The wave is fully nonlinear because of we did not consider any approximation in the physical quantities and we solved the basic equations exactly. This AASPP fulfills the following conditions: the Sagdeev potential and its 1st derivative vanish at $\phi = 0, i.e.\ S(0) = \frac{dS(\varphi)}{d\varphi}|_{\varphi=0}$. The solution of the solitary wave exists for the plasma system, if the second derivative of AASPP (18) less than zero i.e.

$$\frac{d^2 S(\varphi)}{d\varphi^2}|_{\varphi=0} < 0, \quad (20)$$

The satisfied condition for certain Mach number (M), is that it is greater than a minimum value that coincides with the vanishing of the second derivative of AASPP

$$M_{\min} = -\mu_e - \mu_{es}\sigma_{es} + \frac{1}{2\sqrt{3\sigma_1}}\left(\frac{1}{\left(M-\sqrt{3\sigma_1}\right)^2} - \frac{1}{\left(M+\sqrt{3\sigma_1}\right)^2}\right) + \mu_p \frac{1}{2\sqrt{3\sigma_2}}\left(\frac{1}{\left(M-u_{p0}+\sqrt{3\sigma_2/Q}\right)^2} - \frac{1}{\left(-M+u_{p0}+\sqrt{3\sigma_2/Q}\right)^2}\right), \quad (23)$$

The minimum mach number $M_{\min}$ remains real and positive the following two conditions are satisfied

$$\left(\begin{array}{l}\frac{1}{2\sqrt{3\sigma_1}}\left(\frac{1}{\left(M-\sqrt{3\sigma_1}\right)^2} - \frac{1}{\left(M+\sqrt{3\sigma_1}\right)^2}\right) + \\ \mu_p \frac{1}{2\sqrt{3\sigma_2}}\left(\frac{1}{\left(M-u_{p0}+\sqrt{3\sigma_2/Q}\right)^2} - \frac{1}{\left(-M+u_{p0}+\sqrt{3\sigma_2/Q}\right)^2}\right)\end{array}\right) > \mu_e + \mu_{es}\sigma_{es}, \quad (24)$$

4. Numerical solution and discussions

Now, we explore the existence zones of the large amplitude dust-ion-acoustic solitary waves in Jupiter ionosphere. The existence zone is confined between the maximum Mach number $M_{\max}$ given by Eq. (24) and the minimum Mach number $M_{\min}$ given by Eq. (23).

In Fig. (1) we shows the minimum and maximum Mach numbers for different values of μ_p. It is seen that increasing the value of μ_p would lead to make the solitary waves propagate for lower speed until the wave Mach number ≈ 1. The solitary pulse profile is depicted in Fig. (1b), it is obvious that increasing μ_p makes the pulse taller and narrower.

Fig. (2a) shows the maximum Mach number and minimum Mach number for various values of σ_2. It is seen that increasing the value of σ_2 would lead to make the solitary waves propagate for higher speed and the wave becomes far away from the Mach number ≈ 1. The solitary pulse profile is depicted in Fig. (2b), it is obvious that increasing σ_2 makes the pulse taller and narrower. Physically, increasing the temperature pumps more energy in the system that lead to increase the amplitude.

Fig. (3a) shows the maximum Mach number and minimum Mach number for various values of σ_{es}. It is seen that increasing the value of σ_{es} would not to a significant change in the Mach number region and the pulses still exist in supersonic zone. The solitary pulse profile is depicted in Fig. (3b), it is obvious that increasing σ_{es} makes the pulse shorter but the width have not significant change.

Fig. (4a) shows the maximum Mach number and minimum Mach number for various values of μ_{es}. It is seen that increasing the value of μ_{es} would not to move the Mach number region to a supersonic zone. The solitary pulse profile is depicted in Fig. (4b), it is obvious that increasing μ_{es} makes the pulse taller but the width have not significant change.

Fig. (5a) shows the range of minimum and maximum Mach numbers for different values of α. It is seen that increasing the value of α would lead to move the Mach number region to a subsonic zone and the pulses become slower. The solitary pulse profile is depicted in Fig. (5b), it is obvious that increasing α makes the pulse towering and the width becomes narrower.

5. Concluding remarks

Finite/Large amplitude dust-ion-acoustic solitary waves are investigated in a dusty plasma consisting of warm two positive ions, and two Maxwellian electrons, as well as stationary dust grains. We derive an energy equation with convenient Sagdeev psudo-potential. The parametric analysis of the latter are obtained, which suggest the pulses could propagate in our system at certain plasma parameters regions only. The dependence of the pulse features on the parameters μ_p, σ_2, σ_{es}, μ_{es}, and α are examined. It is obvious that this plasma supports only compressive solitary excitation. The present results may be useful in understanding the basic features of dust-ion-acoustic solitary waves in Jupiter environment.

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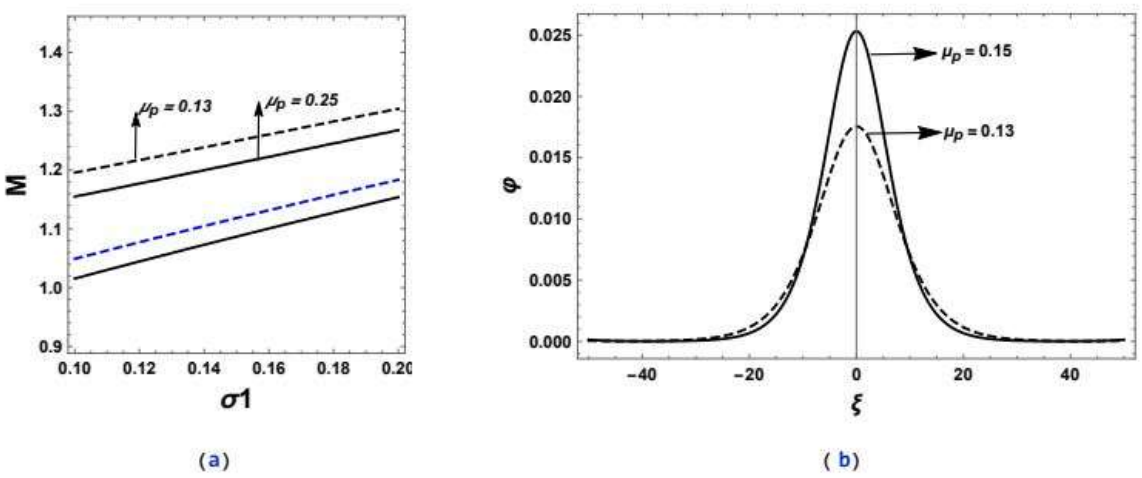

Figure 1: (a) The maximum and minimum Mach numbers M vs σ_1 for different values of μ_p.

(b)The solitary wave profile vs μ_p. Here, $\sigma_2 = 0.19$, $\sigma_{es} = 1.01$, $\mu_{es} = 0.25$, $u_{p0} = 10$, and $\alpha = 0.12$.

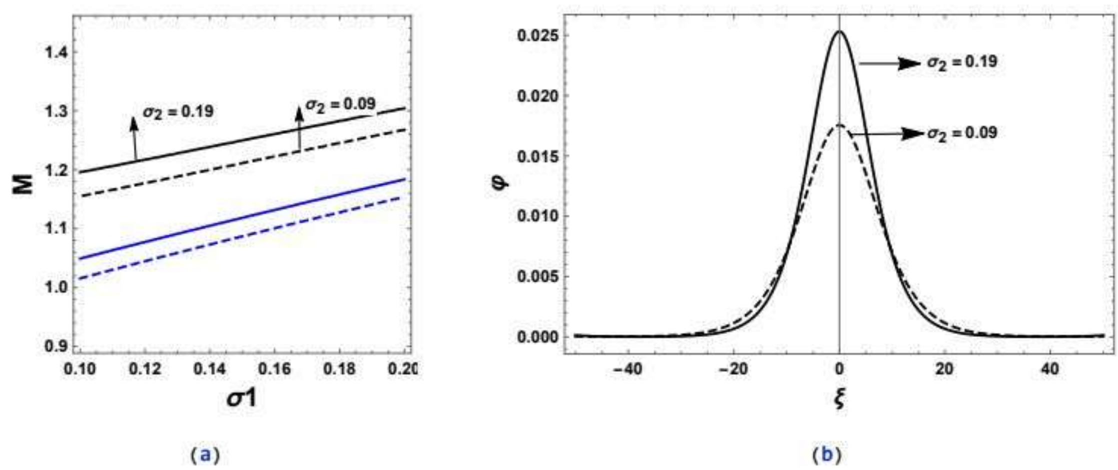

Figure 2: (a) The maximum and minimum Mach numbers M vs σ_1 for different values of σ_2.

(b)The solitary wave profile vs σ_2. Here, $\mu_p = 0.25$, $\sigma_{es} = 1.01$, $\mu_{es} = 0.25$, $u_{p0} = 10$, and $\alpha = 0.12$.

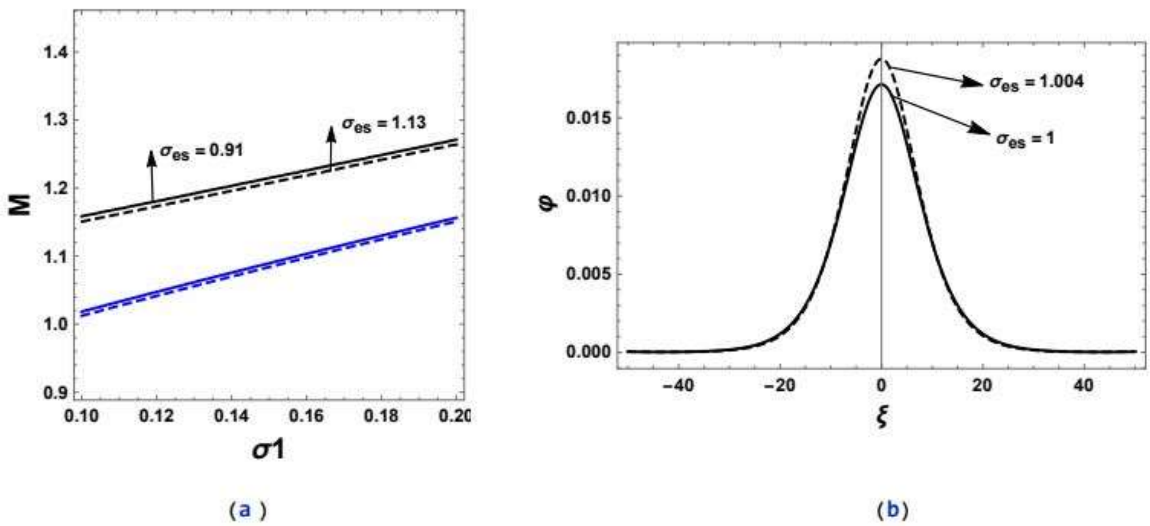

Figure 3: (a) The maximum and minimum Mach numbers M vs σ_1 for different values of σ_{es}. (b)The solitary wave profile vs σ_{es}. Here, $\mu_p = 0.25$, $\sigma_2 = 0.19$, $\mu_{es} = 0.25$, $u_{p0} = 10$, and $\alpha = 0.12$.

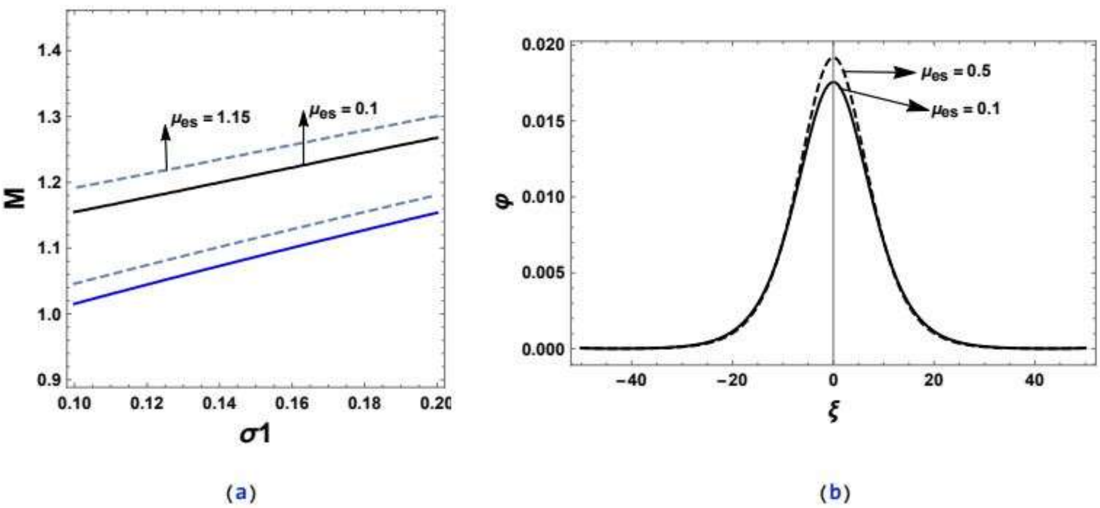

Figure 4: (a) The maximum and minimum Mach numbers M vs σ_1 for different values of μ_{es}. (b)The solitary wave profile vs μ_{es}. Here, $\mu_p = 0.25$, $\sigma_{es} = 1.01$, $\sigma_2 = 0.19$, $u_{p0} = 10$, and $\alpha = 0.12$.

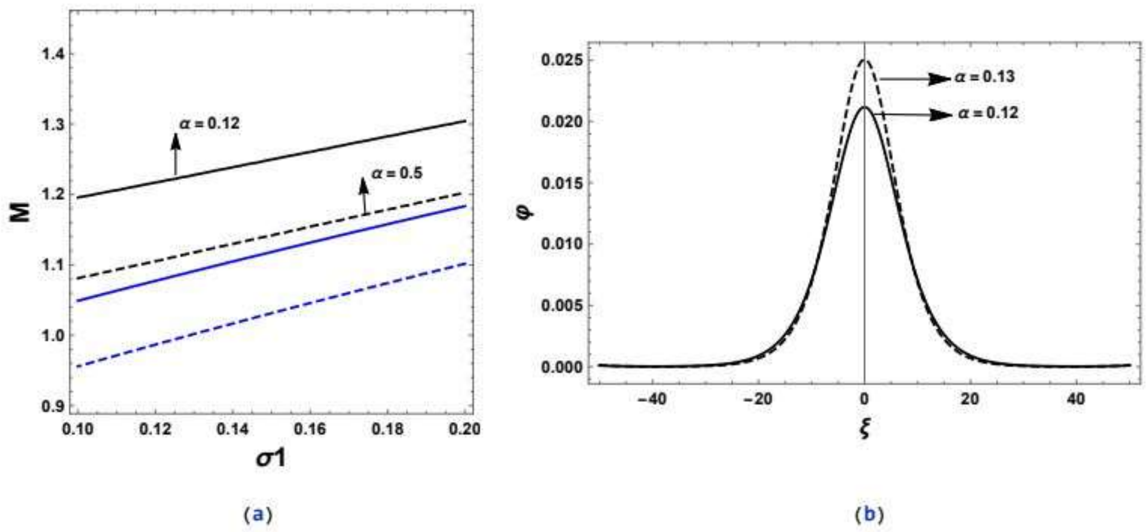

Figure 5: (a) The maximum and minimum Mach numbers M vs σ_1 for different values of α.

(b)The solitary wave profile vs α. Here, $\mu_p = 0.25$, $\sigma_{es} = 1.01$, $\sigma_2 = 0.19$, $u_{p0} = 10$, and $\mu_{es} = 0.25$.