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Topological Approaches for Theoretical Mathematical Morphology

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ABSTRACT

Mathematical morphology was basically introduced to deal with the shapes of images. In this paper, a novel point of view for mathematical morphology via topological concepts is proposed. And, the concepts of set theory and topological spaces will be imported to mathematical morphology. Furthermore, several topological spaces using morphological point of view are presented; by giving definitions for the families of sets that generates each space.

Key Words:

Mathematical morphology; Dilation; Erosion; Topological spaces; Co-topological spaces; Semi-eroded sets; α-eroded sets; pre-dilated sets

1. INTRODUCTION

In 1735, topology developed initially by the German scientist and mathematician Leonhard Euler, among the most important basic concepts in modern mathematics are "topology" and "topological spaces". Topology is basically a branch of mathematics that deals with sets, points and members, within a set and analyzes the relationships among them. In this way, with the help of topology, other concepts such as: interior points, exterior points [2], the notation of α-open and semi-open sets were introduced by Njastad and Levine ([5], [8]).

Mathematical Morphology (MM) introduced by Matheron [6] and Serra ([9], [10]), the mathematical morphology was originally developed to analyze and process binary image using the concepts and operations. Sets in mathematical morphology represented the shapes which were manifested on binary or gray images. The set of all the black pixels in a black and white image, constitutes a complete description of the binary image [11]. We begin the basic idea of our discussion with the binary morphological operations of dilation and erosion. Dilating and eroding are two basic operations of MM. These two operations can make up of some compound operations. An image (set) can be represented by a set of pixels (points - elements). A morphological operation uses two sets, the original image (set) to be analyzed and a structuring element (SE).

In this paper, we will first discuss the concept of morphological topological spaces, more topological structures and based on examples. Also, we will introduce new concepts based on these principles and definitions. Further, their topological properties are studied.

2. Mathematical morphology

In this section, we review the definitions of the classical binary morphological operators as given by Heijmans [4]; which are consistent with the original definitions of the Minkowski addition and subtraction [3]. The basic morphological operations, the dilation and the erosion.

Definition 2.1: Dilation is the morphological transformation which combines two sets using vector addition of set elements. if A and SE are sets, then the dilation of A by SE is the set of all possible vector sums of pairs of elements, one coming from A and one coming from SE. The dilation of A by SE is denoted by:

$$\mathrm{A} \oplus \mathrm{SE} = \delta(A).$$

Definition 2.2: Erosion is the morphological dual to dilation. It is the morphological transformation which combines two sets using the vector subtraction of set elements. If A and SE are two sets. Some image processing people use the name shrink or reduce for erosion. The erosion of A by SE is denoted by:

$$\mathrm{A} \ominus \mathrm{SE} = \varepsilon(A).$$

In the following, we discuss the connection between the mathematical morphology and the topological operators, the proof of the following remarks can be found in [9, 11].

Remark 2.1: The erosion of a set A by structuring element (SE) is the set of all elements of the set which are certainly in A, which is considered as the interior of the set under consideration.

Remark 2.2: The dilation of a set A by a structuring element (SE) is the set of all elements which are possibly in A, and acts as the closure of the set.

Remark 2.3: Erosion and dilation are dual operations: $(\varepsilon(A)\ )^C = \delta(A^C)$; where A^C is the complement of A.

Remark 2.4: The basic morphological operators, erosion and dilation, are non-idempotent [7]; that is unlike the topological operator, the interior and the closure, that is:

$$\varepsilon(\varepsilon A) \subseteq \varepsilon(A), \text{ while } int\ (intA) = int(A)$$

$$\text{and } \delta(A) \subseteq \delta(\delta A), \text{ while } cl(cl\ A) = cl(A).$$

3. Morphological topological spaces

To commence, we give the following definitions for the basic morphological topological operators; namely, the τ_dilation, and the τ_erosion operators. In addition, we discuss some of the properties of these operators.

Definition 3.1: For any set X, and the two subsets $\mathrm{A}, \mathrm{SE} \subseteq \mathrm{X}$, we define the following operators:

1. τ_dilation of A by SE is denoted by: $\delta^{\tau}(A) = A \oplus^{\tau} SE = \{\ x \in X: (\{x\} \cup SE) \cap A \neq \emptyset\}$.
2. τ_erosion of A by SE is denoted by: $\varepsilon^{\tau}(A) = A \ominus^{\tau} SE = \{\ x \in X: (\{x\} \cup SE) \subseteq A\}$.

Where SE is the structure element; the shape of the structure element can be chosen by the user.

Proposition 3.1: For any set X, and the two subsets A, B $\subseteq$ X; we have the following properties:

a) $\varepsilon^{\tau}(A) \subseteq A$,
b) $\varepsilon^{\tau}(X) = X$,
c) $\varepsilon^{\tau}(\emptyset) = \emptyset$,
d) $\varepsilon^{\tau}(\varepsilon^{\tau}A)) \subseteq \varepsilon^{\tau}(A)$,
e) $\varepsilon^{\tau}(A) \cup \varepsilon^{\tau}(B) \subseteq \varepsilon^{\tau}(A \cup B)$,
f) $\varepsilon^{\tau}(A) \cap \varepsilon^{\tau}(B) = \varepsilon^{\tau}(A \cap B)$.

Proof: For any sets A, B and a structure element SE:

The proof of Parts a), b) and c) are obvious from definitions.

(d) Since, $\varepsilon^{\tau}(\varepsilon^{\tau}A)) = \{x \epsilon X: (\{x\} \cup SE) \subseteq \varepsilon^{\tau}(A)\}$, where $\varepsilon^{\tau}(A) \subseteq A$

$= \{x \epsilon X: (\{x\} \cup SE) \subseteq \varepsilon^{\tau}(A) \subseteq A\}$

$\Longrightarrow \{x \epsilon X: (\{x\} \cup SE) \subseteq (A)\} = \varepsilon^{\tau}(A)$.

(e) Since, $\varepsilon^{\tau}(A) \cup \varepsilon^{\tau}(B) = \{x \epsilon X: (\{x\} \cup SE) \subseteq A \ or \ (\{x\} \cup SE) \subseteq B\}$

$\Longrightarrow \{x \epsilon X: (\{x\} \cup SE) \subseteq (A \cup B)\} = \varepsilon^{\tau}(A \cup B)$.

(f) Since, $\varepsilon^{\tau}(A \cap B) = \{x \epsilon X: (\{x\} \cup SE) \subseteq (A \cap B)\}$

$= \{x \epsilon X: (\{x\} \cup SE) \subseteq A \quad and \ \ (\{x\} \cup SE) \subseteq B\}$

$= \varepsilon^{\tau}(A) \cap \varepsilon^{\tau}(B)$.

Proposition 3.2: For any set X, and the two subsets A, B $\subseteq$ X; we have the following properties:

a) $A \subseteq \delta^{\tau}(A)$,
b) $\delta^{\tau}(\emptyset) = \emptyset$,
c) $\delta^{\tau}(X) = X$,
d) $\delta^{\tau}(A) \subseteq \delta^{\tau}\delta^{\tau}(A)$,
e) $\delta^{\tau}(A) \cup \delta^{\tau}(B) = \delta^{\tau}(A \cup B)$,
f) $\delta^{\tau}(A \cap B) \subseteq \delta^{\tau}(A) \cap \delta^{\tau}(B)$,
g) $A \subseteq B \ then \ \delta^{\tau}(A) \subseteq \delta^{\tau}(B)$.

Proof: For any sets A, B and a structure element SE:

The proof of Parts a), b) and c) are obvious from definitions.

(d) Since, $\delta^{\tau}(A) = \{x \epsilon X: (\{x\} \cup SE) \cap A \neq \emptyset\}, where \ A \subseteq \delta^{\tau}(A)$

$\Longrightarrow \{x \epsilon X: (\{x\} \cup SE) \cap \delta^{\tau}(A) \neq \emptyset\} = \delta^{\tau}\delta^{\tau}(A)$.

(e) Since, $\delta^{\tau}(A) \cup \delta^{\tau}(B) = \{x \epsilon X: [(\{x\} \cup SE) \cap A \neq \emptyset] \ or \ [(\{x\} \cup SE) \cap B \neq \emptyset]\}$

$= \{x \epsilon X: (\{x\} \cup SE) \cap (A \cup B) \neq \emptyset\}$

$= \delta^{\tau}(A \cup B)$.

(f) Since, $\delta^{\tau}(A \cap B) = \{x \epsilon X: (\{x\} \cup SE) \cap (A \cap B) \neq \emptyset\}$

$\Longrightarrow \{x \epsilon X: [(\{x\} \cup SE) \cap A \neq \emptyset] \cap [(\{x\} \cup SE) \cap A \neq \emptyset]\}$

$= \delta^{\tau}(A) \cap \delta^{\tau}(B)$.

(g) Since, $\delta^{\tau}(A) = \{x \epsilon X: (\{x\} \cup SE) \cap A \neq \emptyset\}, where \ A \subseteq B$

$\Longrightarrow \{x \epsilon X: (\{x\} \cup SE) \cap B \neq \emptyset\} = \delta^{\tau}(B)$.

4. Some results on morphological topological structures

Definition 4: Let X be a non-empty set, A, B $\subseteq$ X are two subsets of X, and τ_M be a collection of subsets of X. Then the triple structure (X, τ_M, SE) is said to be a morphological topological space if it satisfies the following conditions:

1. The empty set $\emptyset$ and the universe X belong to τ_M,
2. $(A \cup B) \ominus_{\tau} SE \in \tau_M$,
3. $(A \cap B) \ominus_{\tau} SE \in \tau_M$.

The elements of X are the objects, τ_M is the class partitioning of the objects. Hence, τ_M is said to be a morphological topology of X.

Definition 4.2: Let X be a non-empty set, A, B $\subseteq$ X are two subsets of X, then the triple structure $(X, co\ \tau_M, SE)$ is said to be a morphological co-topological space, if it satisfies the following condition:

1. The empty set $\emptyset$ and the universe X belong to $co\ \tau_M$,
2. $(A \cup B) \oplus_{\tau} SE \in co\ \tau_M$,
3. $(A \cap B) \oplus_{\tau} SE \in co\ \tau_M$.

Example 4.1: For the set X = {a, b, c, d}, One can construct the following (Table 1), which apply the τ_dilation, and the τ_erosion operators over several subsets of X using the structure element SE = $\{a, b\}$.

Table 1: Morphological Topological Operators

A	$\delta^{\tau}(A)$	$\varepsilon^{\tau}(A)$	$\delta^{\tau}(\varepsilon^{\tau}(A))$	$\varepsilon^{\tau}(\delta^{\tau}(A))$
{a}	X	∅	∅	X
{b}	X	∅	∅	X
{c}	{c}	∅	∅	∅
{d}	{d}	∅	∅	∅
{a, b}	X	{a, b}	X	X
{a, c}	X	∅	∅	X
{a, d}	X	∅	∅	X
{b, c}	X	∅	∅	X
{b, d}	X	∅	∅	X
{c, d}	{c, d}	∅	∅	∅
{a, b, c}	X	{a, b, c}	X	X
{a, b, d}	X	{a, b, d}	X	X
{b, c, d}	X	∅	∅	X
{a, c, d}	X	∅	∅	X
X	X	X	X	X
∅	∅	∅	∅	∅

We can choose $\tau_M = \{X, \emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, then (X, τ_M, SE) is a morphological topological space, where:

- $(A \cup B) \ominus_{\tau} SE \in \tau_M$,
- $(A \cap B) \ominus_{\tau} SE \in \tau_M$.

OR,

$\tau_M = \{X, \emptyset, \{a,b\}, \{a,b,c\}\}$, then (X, τ_M, SE) is morphological topological space, since:

- $(A \cup B) \ominus_\tau SE \in \tau_M$,
- $(A \cap B) \ominus_\tau SE \in \tau_M$.

Then, $co\tau_M = \{X, \emptyset, \{c\}, \{c,d\}, \{d\}\}$, then $(X, co\tau_M, SE)$ is morphological co-topological space, since:

- $(A \cup B) \oplus_\tau SE \in co\tau_M$,
- $(A \cap B) \oplus_\tau SE \in co\tau_M$.

OR,

$\tau_M = \{X, \emptyset, \{c\}, \{c,d\}\}$, then (X, τ_M, SE) is morphological co-topological space, where:

- $(A \cup B) \oplus_\tau SE \in \tau_M$,
- $(A \cap B) \oplus_\tau SE \in \tau_M$.

Example 4.2: A simple decision information system is shown in the table below (Table 2). There are six objects (patients), four attributes (disease diagnosis) and decision attribute (Flu) with two possible outcomes have been added.

For, the set of patients X= $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ and a structure element SE = $\{E_1, E_4\}$, consider the subsets T, H, N *and* $C \subseteq X$, where:

$T = \{E_1, E_3, E_4\}$, $H = \{E_1, E_2, E_4, E_5\}$, $N = \{E_2, E_4, E_6\}$, $C = \{E_1, E_4, E_6\}$.

Table 2: Decision Information System

Case	Temperature	Headache	Nausea	Cough	Flu
E_1	High	Yes	No	Yes	Yes
E_2	Very-high	Yes	Yes	No	No
E_3	High	No	No	No	No
E_4	High	Yes	Yes	Yes	Yes
E_5	Normal	Yes	No	No	No
E_6	Normal	No	yes	Yes	Yes

$\delta^\tau(T) = \{E_1, E_2, E_3, E_4, E_5, E_6\}$, $\varepsilon^\tau(T) = \{E_1, E_3, E_4\}$,

$\delta^\tau(H) = \{E_1, E_2, E_3, E_4, E_5, E_6\}$, $\varepsilon^\tau(H) = \{E_1, E_2, E_4, E_5\}$,

$\delta^\tau(N) = \{E_1, E_2, E_3, E_4, E_5, E_6\}$, $\varepsilon^\tau(N) = \emptyset$,

$\delta^\tau(C) = \{E_1, E_2, E_3, E_4, E_5, E_6\}$, $\varepsilon^\tau(C) = \{E_1, E_4, E_6\}$.

Remark 4.1: The erosion of a set A by structuring element (SE) is the set of all patients in A that have the same symptoms as the patients in the structure element.

Remark 4.2: The dilation of a set A by a structuring element (SE) is the set of all patients in A that may have the same symptoms as the patients in the structure element.

5. Semi-erosion and semi-dilation topological concepts

In this section, we are proposing new morphological topological structures, namely: morphological semi-topological spaces.

Definition 5.1: Let A be a nonempty set and $A \subseteq X$. Then

1. A is a semi-eroded set if and only if $A \subseteq \delta^\tau(\varepsilon^\tau(A))$.

2. A set $F \subseteq X$ is called a semi-dilated set if the complement of F is semi-eroded set equivalently, F is semi-dilated set if and only if $\varepsilon^{\tau}(\delta^{\tau}(F)) \subseteq F$.

Corollary 5.1: The complement of a semi-eroded set is semi-dilated and the complement of a semi-dilated set is semi-eroded.

Proof :Let $A \subseteq X$ be a semi-eroded set. Then $A \subseteq \delta^{\tau}\ (\varepsilon^{\tau}(A))$

$$\Leftrightarrow A^c \supseteq \left(\delta^{\tau}\ (\varepsilon^{\tau}A)\right)^C$$

$$\Leftrightarrow A^c \supseteq \varepsilon^{\tau}((\varepsilon^{\tau}A)^c)$$

$$\Leftrightarrow A^c \supseteq \varepsilon^{\tau}\delta^{\tau}(A^c).$$

Then, A^c is a semi-dilated set.

Example 5.1: From (Table 1), in Example 4.1, we have:

- The sets {{a, b}, {a, b, c} and {a, b, d}}, are semi-eroded sets,
- The sets $\{\{c, d\}, \{d\}$ and $\{c\}\}$, are semi-dilated sets.

Definition 5.2: For a morphological topological space(X, τ_M, SE);

1. The structure$(X, \varepsilon_S^{\tau}, SE)$ is said to be a morphological semi-topological space, where ε_S^{τ} is a collection of the semi-eroded subsets of X. The collection ε_S^{τ} is called a morphological semi-topology.
2. The structure(X, δ_S^{τ}, SE), is said to be a morphological semi-co-topological space, where δ_S^{τ} is a collection of the semi-dilated subsets of X. The collection δ_S^{τ} is called a morphological semi-co-topology.

Corollary 5.2: The complement of a morphological semi-topology is morphological semi-cotopology.

Example 5.2: From (Table 3), in Example 5.2, we have:

- The collection $\varepsilon_S^{\tau} = \{X, \emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, is a morphological semi-topology,
- The collection $\delta_S^{\tau} = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}\}$, is a morphological semi-co-topology.

Remark 5.1: from the previous example [5.2], one may notice that the deduced semi-dilated and semi-eroded satisfies the axioms of the supra topology given in [1] that is; a sub collection of the power set of x is called a supra topology on X if X belongs to $\boldsymbol{\tau}$ and $\boldsymbol{\tau}$ is closed under arbitrary union. Every element of $\boldsymbol{\tau}$ is called a supra open set of (X, $\boldsymbol{\tau}$) and its complement is called supra closed.

6. α-erosion and α-dilation topological concepts

In this section, we're proposing new morphological topological structures, namely: morphological α-topological spaces

Definition 6.1: Let A be a nonempty set and $A \subseteq X$. Then

1. A is an α-eroded set if and only if $A \subseteq\ \varepsilon^{\tau}(\delta^{\tau}\ (\varepsilon^{\tau}A))$.
2. A set $F \subseteq X$ is called α-dilated set if the complement of F is α-eroded equivalently, F is an α-dilated set if and only if $\delta^{\tau}\ (\varepsilon^{\tau}(\delta^{\tau}(F))) \subseteq F$.

Corollary 6.1 :The complement of α-eroded set is α-dilated and the complement of α-dilated set is α-eroded.

Proof:

Let $A \subseteq X$ be α -eroded set. Then $A \subseteq \varepsilon^{\tau}(\delta^{\tau}(\varepsilon^{\tau}A))$

$\Leftrightarrow A^c \supseteq (\varepsilon^{\tau}(\delta^{\tau}(\varepsilon^{\tau}A)))^c$

$\Leftrightarrow A^c \supseteq \delta^{\tau}\varepsilon^{\tau}(\varepsilon^{\tau}A)^c$

$\Leftrightarrow A^c \supseteq \delta^{\tau}\varepsilon^{\tau}\delta^{\tau}(A^c)$.

Then, A^c is α-dilated set.

Example 6.1: For the sets are given in Example 4.1, we construct the following table:

Table 3: Morphological Topological Operators

A	$\delta^{\tau}(\varepsilon^{\tau}(\delta^{\tau}(A)))$	$\varepsilon^{\tau}(\delta^{\tau}(\varepsilon^{\tau}(A)))$
{a}	X	∅
{b}	X	∅
{c}	∅	∅
{d}	∅	∅
{a, b}	X	X
{a, c}	X	∅
{a, d}	X	∅
{b, c}	X	∅
{b, d}	X	∅
{c, d}	∅	∅
{a, b, c}	X	X
{a, b, d}	X	X
{b, c, d}	X	∅
{a, c, d}	X	∅
X	X	X
∅	∅	∅

Hence, we have that:

- The sets {{a, b}, {a, b, c}, {a, b, d}}, are α-eroded sets,
- The sets $\{\{c,d\},\{d\},\{c\}\}$, are α-dilated sets.

Definition 6.2: For a morphological topological space(X, τ_M, SE);

1. The structure$(X, \varepsilon_{\alpha}^{\tau}, SE)$, is said to be a morphological α-topological space, where $\varepsilon_{\alpha}^{\tau}$ is a collection of the α-eroded subsets of X. The collection $\varepsilon_{\alpha}^{\tau}$ is called a morphological α-topology.
2. The structure$(X, \delta_{\alpha}^{\tau}, SE)$, is said to be a morphological α-co-topological space, where δ_{α}^{τ} is a collection of the α-dilated subsets of X. The collection δ_{α}^{τ} is called a morphological α-co-topology.

Corollary 6.2: The complement of a morphological α-topology is morphological α-co-topology.

Example 6.2: From (Table 3), in Example 5.2, we have:

- The collection $\varepsilon_{\alpha}^{\tau} = \{X, \emptyset, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$, is a morphological α-topology,
- The collection $\delta_{\alpha}^{\tau} = \{X, \emptyset, \{c\}, \{d\}, \{c,d\}\}$, is a morphological α-co-topology.

Remark 6.1: similarity, as stated in remark 5.1, we may obtain that family of morphological α-topology also satisfy the axioms of supra topology.

7. Pre-erosion and pre-dilation topological concepts

In this section, we're proposing new morphological topological structures, namely: morphological pre-topological space.

Definition 7.1: Let A be nonempty set and $A \subseteq X$. Then

1. A is a pre-eroded set if and only if $A \subseteq \varepsilon^{\tau}(\delta^{\tau}(A))$.
2. A set $F \subseteq X$ is called a pre-dilated set if the complement of F is pre-eroded equivalently, F is pre-dilated set if and only if $\delta^{\tau}(\varepsilon^{\tau}(\delta^{\tau}(F))) \subseteq F$.

Theorem 7.1: For any sets $A, B \subseteq X$, the pre-erosion operator, which is monotonicity, satisfies the following properties:

a) $\varepsilon_{\mathcal{P}}(X) = X$,
b) $\varepsilon_{\mathcal{P}}(\emptyset) = \emptyset$,
c) $A \subset B$ *then* $\varepsilon_{\mathcal{P}}(A) \subset \varepsilon_{\mathcal{P}}(B)$,
d) $\varepsilon_{\mathcal{P}}\left(\varepsilon_{\mathcal{P}}(A)\right) \subseteq \varepsilon_{\mathcal{P}}(A)$,
e) $\varepsilon_{\mathcal{P}}(A \cup B) \supset \varepsilon_{\mathcal{P}}(A) \cup \varepsilon_{\mathcal{P}}(B)$,
f) $\varepsilon_{\mathcal{P}}(A \cap B) \subset \varepsilon_{\mathcal{P}}(A) \cap \varepsilon_{\mathcal{P}}(B)$.

Proof: For any sets A, B and a structure element SE:

The proof of (a, b) is obvious from definitions.

(c) Since, $\varepsilon_{\mathcal{P}}(A) = \delta^{\tau}(\varepsilon^{\tau}(A))$, $\varepsilon_{\mathcal{P}}(B) = \delta^{\tau}(\varepsilon^{\tau}(B))$ if $A \subseteq B$

$$\varepsilon^{\tau}(A) \subset \varepsilon^{\tau}(B)$$

$$\delta^{\tau}(\varepsilon^{\tau}(A)) \subset \delta^{\tau}(\varepsilon^{\tau}(B)) \Leftrightarrow \varepsilon_{\mathcal{P}}(A) \subset \varepsilon_{\mathcal{P}}(B).$$

(d) Since, $\varepsilon_{\mathcal{P}}(\varepsilon_{\mathcal{P}}(A) = \left\{x\epsilon X: (\{x\} \cup SE) \subseteq \varepsilon_{\mathcal{P}}(A)\right\} \; and \; \delta^{\tau}(\varepsilon^{\tau}\left(\varepsilon_{\mathcal{P}}(A)\right)) \subseteq \varepsilon_{\mathcal{P}}(A)\big\}, where \; \varepsilon_{\mathcal{P}}(A) \subseteq A$

$$\Longrightarrow \{x\epsilon X: (\{x\} \cup SE) \subseteq A\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(A)) \subseteq (A)\} = \varepsilon_{\mathcal{P}}(A).$$

(e) Since, $\varepsilon_{\mathcal{P}}(A) \cup \varepsilon_{\mathcal{P}}(B)$

$$= \{x\epsilon X: (\{x\} \cup SE) \subseteq A\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(A)) \subseteq A\} \cup \{x\epsilon X: (\{x\} \cup SE) \subseteq B\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(B)) \subseteq B\}$$
$$\Longrightarrow \{x\epsilon X: (\{x\} \cup SE) \subseteq (A \cup B)\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(A \cup B)) \subseteq (A \cup B)\} = \varepsilon_{\mathcal{P}}(A \cup B)$$

(f) Since, $\varepsilon_{\mathcal{P}}(A \cap B) = \{x\epsilon X: (\{x\} \cup SE) \subseteq (A \cap B)\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(A \cap B)) \subseteq (A \cap B)\}$

$$\Longrightarrow \{x\epsilon X: (\{x\} \cup SE) \subseteq (A)\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(A)) \subseteq (A)\} \cap \{x\epsilon X: (\{x\} \cup SE) \subseteq (B)\} \; and \; \delta^{\tau}(\varepsilon^{\tau}(B)) \subseteq (B)\}$$

$$= \varepsilon_{\mathcal{P}}(A) \cap \varepsilon_{\mathcal{P}}(B).$$

Theorem 7.2: For any sets $A, B \subseteq X$, the pre-dilation operator which is monotonicity, satisfies the following properties:

a) $\delta_{\mathcal{P}}(X) = X$,
b) $\delta_{\mathcal{P}}(\emptyset) = \emptyset$,
c) $\delta_{\mathcal{P}}\left(\delta_{\mathcal{P}}(A)\right) = \delta_{\mathcal{P}}(A)$,
d) $A \subseteq B$ *then* $\delta_{\mathcal{P}}(A) \subseteq \delta_{\mathcal{P}}(B)$,
e) $\delta_{\mathcal{P}}(A \cup B) = \delta_{\mathcal{P}}(A) \cup \delta_{\mathcal{P}}(B)$,
f) $\delta_{\mathcal{P}}(A \cap B) = \delta_{\mathcal{P}}(A) \cap \delta_{\mathcal{P}}(B)$.

Proof: For any sets A, B and a structure element SE:

The proof of (a, b) is obvious from definitions.

(c) Since, $\delta_{\mathcal{P}}(A) = \{x \epsilon X: (\{x\} \cup SE) \cap A \neq \emptyset \; and \; \mathrm{A} \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(A)\}, where \; A \subseteq \delta^{\tau}(A)$

$$\Longrightarrow \{x \epsilon X: (\{x\} \cup SE) \cap \delta^{\tau}(A) \neq \emptyset \; and \; \delta^{\tau}(A) \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(\delta^{\tau}(A))\} = \delta_{\mathcal{P}}\delta_{\mathcal{P}}(A).$$

(d) Since, $\delta_{\mathcal{P}}(A) = \; \varepsilon^{\tau}(\delta^{\tau}(A)), \delta_{\mathcal{P}}(B) = \; \varepsilon^{\tau}(\delta^{\tau}(B))$, if $A \subseteq B$

$$\delta^{\tau}(A) \subseteq \delta^{\tau}(B)$$

$$\varepsilon^{\tau}(\delta^{\tau}(A)) \subseteq \varepsilon^{\tau}(\delta^{\tau}(B)) \Leftrightarrow \delta_{\mathcal{P}}(A) \subseteq \; \delta_{\mathcal{P}}(B).$$

(e) Since, $\delta_{\mathcal{P}}(A) \cup \delta_{\mathcal{P}}(B)$

$$= \{x \epsilon X: (\{x\} \cup SE) \cap A \neq \emptyset \; and \; \mathrm{A} \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(A)\} \cup \{x \epsilon X: (\{x\} \cup SE) \cap B \neq \emptyset \; and \; \mathrm{B} \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(B)\}$$

$$= \{x \epsilon X: (\{x\} \cup SE) \cap (A \cup B) \neq \emptyset \; and \; (\mathrm{A} \cup B) \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(A \cup B)\} \; = \delta_{\mathcal{P}}(A \cup B).$$

(f) Since, $\delta_{\mathcal{P}}(A \cap B) = \{x \epsilon X: (\{x\} \cup SE) \cap (A \cap B) \neq \emptyset \; and \; (\mathrm{A} \cap B) \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(A \cap B)\}$

$$= \{x \epsilon X: (\{x\} \cup SE) \cap A \neq \emptyset \; and \; \mathrm{A} \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(A)\} \cap \{x \epsilon X: (\{x\} \cup SE) \cap B \neq \emptyset \; and \; \mathrm{B} \subseteq \; \varepsilon^{\tau}(\delta^{\tau}(B)\}$$

$$= \delta_{\mathcal{P}}(A) \cap \; \delta_{\mathcal{P}}(B).$$

Example 7.1: From (Table 1), in Example 4.1, we have:

- The sets ({a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}), are pre-eroded sets,
- The sets ({a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {b, c, d}, {a, c, d}), are pre-dilated sets.

Definition 7.2: For a morphological topological space(X, τ_M, SE);

1. The structure$(X, \varepsilon_{\mathcal{P}}^{\tau}, SE)$, is said to be a morphological pre-topological space, where $\varepsilon_{\mathcal{P}}^{\tau}$ is a collection of the pre-eroded subsets of X. The collection $\varepsilon_{\mathcal{P}}^{\tau}$ is called a morphological pre-topology.
2. The structure$(X, \delta_{\mathcal{P}}^{\tau}, SE)$, is said to be a morphological pre-co-topological space, where $\delta_{\mathcal{P}}^{\tau}$ is a collection of the pre-dilated subsets of X. The collection $\delta_{\mathcal{P}}^{\tau}$ is called a morphological pre-co-topology.

Corollary 7.1: The complement of a morphological pre-topology is morphological pre-co-topology.

Example 7.2: From (Table 3), in Example 5.2, we have:

- The collection
  $\varepsilon_{\mathcal{P}=}^{\tau}\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$,
  is a morphological pre-topology,
- The collection
  $\delta_{\mathcal{P}}^{\tau} = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$,
  is a morphological pre-co-topology.

Remark 7.1: similarity, as stated in remark 5.1, we may obtain that family of morphological pre-topology also satisfy the axioms of supra topology.

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7. Conclusion

This work proposed new definitions of mathematical morphology using the concepts of set theory and topology. Also, we presented several topological spaces using morphological point of view; namely, morphological topological spaces, morphological co-topological spaces, morphological semi-topological spaces, morphological α-topological spaces and morphological pre-topological spaces. Furthermore, we introduced the definitions for the families of sets that generate each space, such as: eroded sets, dilated sets, semi-eroded sets, semi- dilated sets, α-eroded sets, α- dilated sets, pre- eroded sets and pre-dilated sets.

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