Alfarama Journal of Basic & Applied Sciences

Faculty of Science Port Said University

https://ajbas.journals.ekb.eg
ajbas@sci.psu.edu.eg
http://sci.psu.edu.eg/en/

ISSN 2682-275X

July 2022, Volume 3 Issue II

DOI: 10.21608/AJBAS.2022.106505.1077

Submitted: 25-11-2021
Accepted: 12-01-2022

Pages: 315-334

Estimation to the Parameters of Truncated Weibull Rayleigh Distribution under Progressive type-II Censoring Scheme with an application to lifetime data

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ABSTRACT

In this paper, the truncated Weibull Rayleigh distribution parameters are estimated using progressive type-II censoring. Maximum-likelihood estimates (MLEs) and Bayesian estimates have been used as point estimates for these parameters. The asymptotic distribution of MLEs is used to create approximate confidence intervals (ACIs) for unknown parameters, the reliability function, and the hazard function. In addition, Bayesian estimates can be used to obtain symmetrical and asymmetric loss functions such as squared error and Linear Exponential Loss Function (LINEX). The Markov chain Monte Carlo (MCMC) technique is used to derive unknown parameter estimates via Bayes estimation. An empirical study of the flexibility of the truncated Weibull Rayleigh (TW-R) distribution using a real data set. The proposed distribution has been sufficiently improved to fail real data from a model Windshield aircraft. As a result, the survival and risk functions were approximated to extrapolate data that contributed to the analysis of the Windshield model of aircraft failure under the progressive type II censoring scheme.

Keywords:

Truncated Weibull Rayleigh distribution, progressive type-II censoring, maximum likelihood estimation, Bayesian estimation.

1. INTRODUCTION

The failure analysis of a Windshield aircraft model with multiple subcomponents required an understanding of their failure behavior, which can be accomplished by conducting a lifetime test on these components. The failure rate is the probability of failing this test at any given time. If a part, component, or system is not working well, it can be upgraded or repaired, and the aircraft's condition can be improved to some level. When objects deteriorate, maintenance might be condition-based. Periodic inspection is a major maintenance component, and losing the ability to detect items with a higher probability of failure during operation can have serious consequences. This significant change is an object storage maintenance process that includes extensive aircraft downtime, corrosion and cracks structural sampling, detailed system tests, and the replacement of worn components. The Federal Aviation Administration maintains the safety problems an aircraft is experienced during operating in the Service Difficulty Reports (SDRs) in the United States (US). The SDR database was evaluated as a possible source of information [1]. In a comparison of failure rates [2], identified significant factors in explaining the differences in the rate of SDR's across carriers, so we restore to applied progressive type-II censoring data. During reliability analysis, experiments are frequently ended before all units fail due to cost and time constraints. Only a component of the sample offers failure information in these circumstances, and all units that have failed have only partial data. Component and unit failure as the primary structure of industrial and mechanical engineering operating systems has been studied by statisticians for a long time. Their research focuses on observing supervising units, recording the units' lifetimes, and using statistical inference methods to analyze the data. Failure data should be fitted into an appropriate parametric statistical distribution in order to identify the parameters of that distribution that lead to estimation of reliability and hazard functions. Knowing the structure of two vital functions, reliability and hazard functions, allows statisticians to predict with a high level of confidence 95% and make the best decision on the survival factor or hazard factor of such models. Some statistical inference approaches such as maximum likelihood and Bayesian methods are applied to estimating the parameters of TW-R model under progressive type-II censoring scheme using real data sets [3] represents failure times of 63 aircraft Windshield. A large aircraft's windshield is a complicated piece of equipment formed of several layers of different materials. The windshield comprises a highly robust outer layer and a heated layer below it. Failures in these objects are not structural failures. The non-structural outer ply or failure of the heating system, on the other hand, usually results in damage or deflection**.** These failures do not result in aircraft damage, but they do necessitate the replacement of the windshield. We consider the data on failure and service times given in table1 for a specific model windshield, these data were recently [4]. The data is made up of 153 observations, with 88 of them being classed as failed windshields and 65 being windshield service times that did not fail when observed. A complicated aircraft is made up of many interconnected parts, systems, and components. The design of electrical and mechanical systems has a long life expectancy, which is measured in hours. As aircraft and systems get older and more used, they gradually decline until they can no longer perform the functions for which they were designed. To put it another way, the system fails. Censoring is a common phenomenon in life testing. Assessments that are censored come in a variety of forms. Type II censoring is one of the most regularly censored tests. Using type II censoring can save time and money. However, if product lifetimes are very long, the censored life test type II's experimental time may still be too long. A generalization of Type-II censoring is progressive Type-II censoring. [5] presented a life test in which the experimenter might organize the test units into many sets, each as an assembly of test units, and then run all of the test units at the same time until the first failure in each group occurs. Under the first failure, such an estimation of the Gompertz distribution's parameters -sampling plan that is censored [6], statistical inference about the shape parameter of the Burr type XII distribution under the failure-censored sampling plan [7] and Estimation of lifetime parameters of the modified extended exponential distribution with application to a mechanical model [8].

When exact survives only reach specific levels in an experiment and only a portion of the test units are known for their exact lifetime, censoring is used. There are several types of censored tests available. One of the most popular censoring schemes is Type II censoring. Type II censoring is tested on a total of n units. Instead, the test is stopped when the m^{th} $(1 \leq m \leq n)$ unit failure. We use a progressive type II censoring method in our research and evaluate the entire system's reliability and hazard functions using the data obtained. The two most common censoring techniques are Type I and Type II. However, because some experimental units are costly and extremely accurate, the number of test units and the time required to experiment with these units must be decreased. The progressive type II censoring system satisfies the demand for good estimators with a short lifetime experiment and the preservation of selected experimental units from failure. A progressive type II censoring method can be defined like thus. First, the experimenter tests the lifetime with n separate and identical units. When the first failure occurs, say at time $t_{(1)}$, r_1 units are randomly removed from remaining $n - 1$ surviving units. When the second failure occurs at time $t_{(2)}$, r_2 units are randomly removed from remaining $n - r_1 - 2$ surviving units. This experiment terminates when the m^{th} failure occurs at time t_m, and $r_m = n - m - \sum_{i=0}^{m-1} r_i$.

An experimenter can avoid the life-testing process altogether, saving time and money for all n items. As a result, the test is considered censored, with the collected data being the exact failure times for those units that failed and the run times for those that did not fail. When the loss of life test units is unavoidable at times other than termination, type-II censoring generalization is useful. The monograph can be used to learn about the theory, methods, and applications of progressive censoring [9] and the survey paper [10].

In statistical analysis, a variety of distributions are selected to represent data sets. In recent years, new distributions, such as the new distribution, have become more flexible in modelling real data in some well-known distribution families. The first step in data modelling was to combine some distributions with each other in some way, [11] was introduced a new family of continuous distributions based on the Truncated Weibull (TW) generating family, called the truncated Weibull Rayleigh distribution. The TW-R distribution provides more flexible model. The cdf and pdf of the truncated Weibull-Rayleigh distribution are given, respectively, by

$$F_{TW-R}(x,\Psi) = A\left[1-e^{-\alpha\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta}}\right] \quad (1)$$

and

$$f_{TW-R}(x,\Psi) = A\,\alpha\,\beta\,\sigma^{-2}\,x\,e^{\frac{-x^2}{2\sigma^2}}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta-1} e^{-\alpha\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta}}, \quad (2)$$

where $A=(1-e^{-\alpha})^{-1}$, σ and β are the shape parameters and α is the scale parameter.

The reliability function is the complement of the cumulative distribution function. If modeling the time to fail, the cumulative distribution function represents the probability of failure.

$$S_{TW-R}(x,\Psi) = 1 - F_{TW-R}(x,\Psi)$$

$$=1-A\left[1-e^{-\alpha\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta}}\right] \quad (3)$$

and

$$h_{TW-R}(x,\Psi) = \frac{f_{TW-R}(x,\Psi)}{S_{TW-R}(x,\Psi)}$$

$$= \frac{A\,\alpha\,\beta\,\sigma^{2}\,x\,e^{\frac{-x^2}{2\sigma^2}}\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta-1} e^{-\alpha\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta}}}{1-A\left[1-e^{-\alpha\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)^{\beta}}\right]}. \quad (4)$$

[11] introduced a new distribution called the truncated Weibull Rayleigh distribution (TW-R), Its characterization and statistic characteristics are acquired, such as reliability, hazard function, reverse hazard function, cumulative hazard function, quantile function, r^{th} moment, incomplete moments, Rényi and q entropies, and statistic order. The estimation parameters are implemented using the maximum estimation method and derived from the Fisher information data matrix. The flexibility of the new model in the modeling of windshield lifetime data for aircraft.

The following paper is arranged. Maximum Likelihood estimates of α,β and σ have been obtained in Section 2. For various loss functions, such as squared error and LINEX in Section 3, the Bayes estimates of $\alpha,\beta,\sigma,S(t)$ and $h(t)$ are derived. Real data sets were analyzed and simulation studies were carried out to analyze the properties of the various estimators developed in Section 4 of this paper. Lastly, Section 5 presents conclusions.

2. MAXIMUM-LIKELIHOOD ESTIMATION

In this section, we study TW-R parameters estimating problems using the maximum-likelihood estimators for progressive type-II-censored samples. Based on the observed sample $x_1<\cdots<x_m$ from a progressive Type-II censoring scheme, $R = (R_1, R_2, \dots, R_m)$. The likelihood function can be written as (see [9])

$$L(data|\alpha,\beta,\sigma) = C\prod_{i=0}^{m} f(x_{i:m:n}|\alpha,\beta,\sigma)[1-F(x_{i:m:n}|\alpha,\beta,\sigma)]^{R_i}, \tag{5}$$

where C is a constant which does not depend on the parameters and it is defined by $C = n(n-R_1-1)\dots\left(n-\sum_{i=1}^{m-1}(R_i+1)\right)$, $f(x_{i:m:n}|\alpha,\beta,\sigma)$ is pdf in Eq. (1) and $F(x_{i:m:n}|\alpha,\beta,\sigma)$ is the cdf of x in Eq. (2). So, the joint probability density function. By using Eq. (4) is given by,

$$L(data|\alpha,\beta,\sigma) = C(1-e^{-\alpha})^{-m}\,\alpha^m\beta^m\,(\sigma)^{-2m}e^{\frac{-\sum_{i=1}^{m}x_i^2}{2\sigma^2}}\prod_{i=1}^{m}x_i\prod_{i=1}^{m}\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta-1}e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}$$
$$\prod_{i=1}^{m}\left[1-(1-e^{-\alpha})^{-1}\left(1-e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right]^{R_i}. \tag{6}$$

To obtain the normal equations for the unknown parameters, we differentiate Eq. (6) partially with respect to the parameters α, β and σ and equate them to zero. The estimators for α, β and σ can be obtained as the solution of the following equations.

$$\frac{\partial\ell}{\partial\alpha} = \frac{-me^{-\alpha}}{(1-e^{-\alpha})} + \frac{m}{\alpha} - \sum_{i=0}^{m}\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}$$
$$+\sum_{i=1}^{m}R_i\frac{e^{-\alpha}(1-e^{-\alpha})^{-1}\left(1-e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}\right)+\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}}{(1-e^{-\alpha})-\left(1-e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}\right)},$$

$$\frac{\partial\ell}{\partial\beta} = \frac{m}{\beta} + \sum_{i=0}^{m}\log\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right) - \alpha\sum_{i=1}^{m}\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}\log\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}$$
$$-\alpha\sum_{i=1}^{m}R_i\frac{\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}\log\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}}{(1-e^{-\alpha})-\left(1-e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}\right)}$$

and

$$\frac{\partial \ell}{\partial \sigma}=\frac{-2m}{\sigma}+\frac{1}{\sigma^3}\sum_{i=0}^{m} x_i^2-\frac{(\beta-1)}{\sigma^3}\sum_{i=1}^{m}\frac{x_i^2 e^{\frac{-x_i^2}{2\sigma^2}}}{\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)}+\frac{\alpha\beta}{\sigma^3}\sum_{i=1}^{m} x_i^2\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta-1}$$

$$+\frac{\alpha\beta}{\sigma^3}\sum_{i=1}^{m} R_i\frac{x_i^2 e^{\frac{-x_i^2}{2\sigma^2}}\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}}{(1-e^{-\alpha})-\left(1-e^{-\alpha\left(1-e^{\frac{-x_i^2}{2\sigma^2}}\right)^{\beta}}\right)}.$$

3. BAYSESIAN ESTIMATOR

In this section, for the unknown parameters α,β and σ, a Bayesian estimate is given beside the squared error loss function. The parameters α,β and σ are assumed to be independent from the gamma distributions. This is done accordingly

$$\begin{cases}\pi_1(\alpha)\propto \alpha^{a_1-1}e^{-b_1\alpha} & ,\alpha>0, \quad a_1>0, \quad b_1>0\,,\\ \pi_2(\beta)\propto \beta^{a_2-1}e^{-b_2\beta} & ,\beta>0, \quad a_2>0, \quad b_2>0\,,\\ \pi_3(\alpha)\propto \sigma^{a_3-1}e^{-b_3\sigma} & ,\sigma>0, \quad a_3>0, \quad b_3>0\,,\end{cases} \tag{7}$$

where the hyper-parameters a_i and $b_i, i=1,2,3$ are supposed to be known, and are chosen to represent a prior assumption about the unknown parameters. The posterior distribution of the parameters α,β and σ denoted by $\pi^*(\alpha,\beta,\sigma \mid \text{data})$ up to proportionality can be obtained by combining the likelihood function Eq. (6) with the prior via Bayes' theorem and it can be written as

$$\pi^*(\alpha,\beta,\sigma \mid \text{data})=\frac{\pi_1(\alpha)\,\pi_2(\beta)\,\pi_3(\sigma)\,L(\alpha,\beta,\sigma \mid \text{data})}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\alpha)\,\pi_2(\beta)\,\pi_3(\sigma)\,L(\alpha,\beta,\sigma \mid \text{data})d\alpha d\beta d\sigma}. \tag{8}$$

A common loss function is the squared error-loss function (SEL), an asymmetric loss function that assigns equal losses to overestimation and underestimating. If φ is the parameter to be estimated by an estimator $\hat{\varphi}$, then the square error loss function is defined as

$$L(\varphi,\hat{\varphi})=(\hat{\varphi}-\varphi)^2. \tag{9}$$

Therefore, the Bayes estimate of any function of α,β and σ, say $g(\alpha,\beta,\sigma)$ under the SEL we can be obtained as

$$\hat{g}_{BS}(\alpha,\beta,\sigma \mid data)=E_{\alpha,\beta,\sigma|data}(g(\alpha,\beta,\sigma)), \tag{10}$$

where

$$E_{\alpha,\beta,\sigma|data}\big(g(\alpha,\beta,\sigma)\big)=\frac{\int_0^\infty\int_0^\infty\int_0^\infty \big(g(\alpha,\beta,\sigma)\big)\,\pi_1(\alpha)\,\pi_2(\beta)\,\pi_3(\sigma)\,L(\alpha,\beta,\sigma \mid \text{data})d\alpha d\beta d\sigma}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\alpha)\,\pi_2(\beta)\,\pi_3(\sigma)\,L(\alpha,\beta,\sigma \mid \text{data})d\alpha d\beta d\sigma}. \tag{11}$$

Because of the complex structure of the probability function, the several integrals cannot be determined analytically. As a result, the author proposes that samples be generated from the joint posterior density function using the MCMC approximation approach and used to compute Bayes, and estimates, as well as any other aspect of such samples such as S(t), h(t), and the formation of credible intervals. We consider Gibbs within the Metropolis sampler to apply the MCMC methodology, which requires that the entire set of posterior distribution conditional be derived. The joint posterior to the proportionality can be written as from Eq. (8).

$$\pi^*(\alpha,\beta,\sigma \mid \text{data}) = \alpha^{a_1+m-1}\beta^{a_2+m-1}\sigma^{a_3-2m-1}\, e^{-b_1\alpha-b_2\beta-b_3\sigma}\left(1-e^{-\alpha}\right)^{-m} \times e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}}\prod_{i=1}^m x_i \prod_{i=1}^m\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}} \times \prod_{i=1}^m\left(\left(1-e^{-\alpha}\right)^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i}. \quad (12)$$

The full conditionals for α,β and σ can be written, up to proportionality, as

$$\pi^*{}_1(\alpha \mid \beta,\sigma, \text{data}) \propto \alpha^{a_1+m-1}e^{-b_1\alpha}\left(1-e^{-\alpha}\right)^{-m} e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}}\prod_{i=1}^m x_i \times \prod_{i=1}^m\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}} \times \prod_{i=1}^m\left(\left(1-e^{-\alpha}\right)^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i}, \quad (13)$$

$$\pi^*{}_2(\beta \mid \alpha,\sigma, \text{data}) \propto \beta^{a_2+m-1}e^{-b_2\beta}e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}}\prod_{i=1}^m x_i \prod_{i=1}^m\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} \times e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}} \prod_{i=1}^m\left(\left(1-e^{-\alpha}\right)^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i} \quad (14)$$

and

$$\pi^*{}_3(\sigma \mid \alpha,\beta, \text{data}) \propto \sigma^{a_3-2m-1}e^{-b_2\sigma}e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}}\prod_{i=1}^m x_i \prod_{i=1}^m\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} \times e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}} \prod_{i=1}^m\left(\left(1-e^{-\alpha}\right)^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i}. \quad (15)$$

3.1 Symmetric Bayes Estimation: The inadequacy of difference-based loss functions, such as the squared error loss, in recent statistical literature. There have been different alternative loss functions suggested, the most well-known [12] normalized squared loss function. The posterior mean for the SEL function is the parameter estimator (symmetric). Therefore, the Bayes estimates α,β and σ are obtained when compared with the loss function are obtained as, respectively,

$$\hat{\alpha}_{SB} = E(\alpha \mid \beta, \sigma\,, \text{data}) = K^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \alpha^{a_1+m-1} e^{-b_1\alpha} \left(1-e^{-\alpha}\right)^{-m}$$

$$\times\, e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}} \prod_{i=1}^m x_i \prod_{i=1}^m \left(1 - e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta}$$

$$\times \prod_{i=1}^m \left(\left(1-e^{-\alpha}\right)^{-1} \left(1 - e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta} \right) \right)^{R_i} d\alpha d\beta d\sigma, \qquad (16)$$

$$\hat{\beta}_{SB} = E(\beta \mid \alpha, \sigma\,, \text{data}) = K^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \beta^{a_2+m-1} e^{-b_2\beta}\, e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}} \prod_{i=1}^m x_i$$

$$\times \prod_{i=1}^m \left(1 - e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta}$$

$$\times \prod_{i=1}^m \left(\left(1-e^{-\alpha}\right)^{-1} \left(1 - e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta} \right) \right)^{R_i} d\alpha d\beta d\sigma, \qquad (17)$$

$$\hat{\sigma}_{SB} = E(\sigma \mid \alpha, \beta\,, \text{data}) = K^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \sigma^{a_3-2m-1} e^{-b_3\sigma}\, e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}} \prod_{i=1}^m x_i$$

$$\times \prod_{i=1}^m \left(1 - e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta}$$

$$\times \prod_{i=1}^m \left(\left(1-e^{-\alpha}\right)^{-1} \left(1 - e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta} \right) \right)^{R_i} d\alpha d\beta d\sigma, \qquad (18)$$

where

$$K = \int_0^\infty \int_0^\infty \int_0^\infty \pi(\alpha, \beta, \sigma) \;\; L(\alpha, \beta, \sigma \mid \text{data})\, d\alpha d\beta d\sigma$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha^{a_1+m-1} \beta^{a_2+m-1} \sigma^{a_3-2m-1}\, e^{-b_1\alpha - b_2\beta - b_3\sigma} \left(1-e^{-\alpha}\right)^{-m}$$

$$\times\, e^{-\frac{\sum_{i=1}^m x_i^2}{2\sigma^2}} \prod_{i=1}^m x_i \prod_{i=1}^m \left(1 - e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^m -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^\beta}$$

$$\times \prod_{i=1}^{m}\left((1-e^{-\alpha})^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i} d\alpha d\beta d\sigma. \tag{19}$$

3.2 A Symmetric Bayes Estimation: In some applications, a symmetric loss function may be ideal. Many researchers have lately identified asymmetric loss functions for reliability and lifetime tests. One of the most popular a symmetric loss functions is linear-exponential (LINEX) loss function which [13]. It used in several papers, for example, [14], [15], [16] and [17]. This function is approximately linearly on one side and rises approximately to zero on the other side. The Bayes estimates of α, β and σ against a symmetric loss function are, respectively, obtained as

$$\hat{\alpha}_{LB} = \frac{1}{c}\, log\big(E\,(e^{-c\alpha}|\text{ data})\big),\; c \neq 0,$$

$$E\,(e^{-c\alpha}|\text{ data}) = K^{-1}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \alpha^{a_1+m-1} e^{-\alpha(c+b_1)} (1-e^{-\alpha})^{-m}$$

$$\times e^{-\frac{\sum_{i=1}^{m} x_i^2}{2\sigma^2}} \prod_{i=1}^{m} x_i \prod_{i=1}^{m}\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^{m} -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}$$

$$\times \prod_{i=1}^{m}\left((1-e^{-\alpha})^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i} d\alpha d\beta d\sigma, \tag{20}$$

$$\hat{\beta}_{LB} = \frac{1}{c}\, log\left(E\left(e^{-c\beta}|\text{ data}\right)\right),\; c \neq 0,$$

where

$$E\left(e^{-c\beta}|\text{ data}\right) = K^{-1}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \beta^{a_2+m-1} e^{-\beta(c+b_2)}\, e^{-\frac{\sum_{i=1}^{m} x_i^2}{2\sigma^2}} \prod_{i=1}^{m} x_i$$

$$\times \prod_{i=1}^{m}\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^{m} -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}$$

$$\times \prod_{i=1}^{m}\left((1-e^{-\alpha})^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i} d\alpha d\beta d\sigma, \tag{21}$$

$$\hat{\sigma}_{LB} = \frac{1}{c}\, log\big(E\,(e^{-c\sigma}|\text{ data})\big),\; c \neq 0,$$

where

$$E\,(e^{-c\sigma}|\text{ data}) = K^{-1}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \sigma^{a_3-2m-1} e^{-\sigma(cb_3)}\, e^{-\frac{\sum_{i=1}^{m} x_i^2}{2\sigma^2}} \prod_{i=1}^{m} x_i$$

$$\times \prod_{i=1}^{m}\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta-1} e^{\sum_{i=1}^{m} -\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}$$

$$\times \prod_{i=1}^{m}\left((1-e^{-\alpha})^{-1}\left(1-e^{-\alpha\left(1-e^{-\frac{x_i^2}{2\sigma^2}}\right)^{\beta}}\right)\right)^{R_i} d\alpha d\beta d\sigma. \qquad (22)$$

Bayes' estimation of α, β and σ cannot be analytically computed. Therefore, we are proposing the MCMC technique to produce posterior distribution samples, and then to calculate Bayes' estimates of α, β and σ and for TW-R(α, β, σ) distribution under progressive type II censored. Gibbs sampling and more general Metropolis-in-Gibbs samplers are an essential subclass of MCMC methods.

4. THE METROPOLIS-HASTING-HASTINGS-WITHIN-GIBBS SAMPLING

[18] analyzed the state calculations equation by fast computer machines, the algorithm called Metropolis Hastings [12]. M-H provides a replacement means by evaluating Bayesian inference and is frequently used to simulate data from a given posterior distribution using an arbitrary proposal distribution. In MCMC technology, the Gibbs and Metropolis-Hastings algorithms are used to generate samples from posterior distributions and compute Bayes estimates. When determining the marginal distributions of interest parameters is complicated, but the conditional distributions on and parameter have good shapes given all the other parameters, the Gibbs sampler is preferable. Conditional distributions can be easily simulated if standard parameters are used. However, because generating samples from complete conditions that correspond to the joint posterior is complicated, we propose combining Metropolis-Hastings with those that are completely a Gibbs chain cycle. However, because generating samples from complete conditions that correspond to the joint posterior is difficult, we recommend combining Metropolis-Hastings with those that are completely a Gibbs chain cycle. More information on this method, see [19], [20], [21] and [22]. Thus utilizing the concept of Gibbs sampling procedure as mentioned above, generate sample from the posterior density function under the assumption that parameter α, β and σ have independent prior pdf . To corporate this technique we consider full conditional posterior densities of α, β and σ, in Eq's (13),(14) and (15).

It can be easily seen that the conditional posteriors of α, β and σ in Eq's (13), (14) and (15), see [23]. As a result, gibbs sampling is not an easy way for MCMC implementation; instead, Metropolis-Hasting (M-H) sampling is required. Because of this, the hybrid M-H stage algorithm for updating α, β and σ is given below in Eq's (13), (14) and (15). We began using the MLEs for α, β and σ to run the Gibbs sampler algorithm. We draw samples from several complete conditions using the latest values of all other variables without achieving a systematic convergence pattern. The following steps now illustrate the Metropolis-Hastings algorithm method in the sampling process of Gibbs:

1) Start with initial guess $(\alpha^{(0)}, \beta^{(0)}, \sigma^{(0)})$

2) Set $j = 1$

3) Excuting the following M-H algorithm, generate $\alpha^{(j)}, \beta^{(j)}$ and $\sigma^{(j)}$ from

$$\pi_1^*\left(\alpha^{(j-1)} \mid \beta^{(j-1)}, \sigma^{(j-1)}, \text{data}\right), \pi_2^*\left(\beta^{(j-1)} \mid \alpha^{(j)}, \sigma^{(j-1)}, \text{data}\right), \pi_3^*\left(\sigma^{(j-1)} \mid \alpha^{(j)}, \beta^{(j)}, \text{data}\right).$$

With the normal proposal distributions

$$N\left(\alpha^{(j-1)}, var(\alpha)\right), N(\beta^{(j-1)}, var(\beta)) \ and \ \ N(\sigma^{(j-1)}, var(\sigma)).$$

i. Generate a proposal α^* from $N\left(\alpha^{(j-1)}, var(\alpha)\right)$, β^* from $N\left(\beta^{(j-1)}, var(\beta)\right)$ and σ^* from $N\left(\sigma^{(j-1)}, var(\sigma)\right)$.

ii. Evaluate the acceptance probabilities

$$\eta_\alpha = \left[1, \frac{\pi_1^*\left(\alpha^* \mid \beta^{(j-1)}, \sigma^{(j-1)}, \text{data}\right)}{\pi_1^*(\alpha^{(j-1)} \mid \beta^{(j-1)}, \sigma^{(j-1)}, \text{data})}\right],$$

$$\eta_\beta = \left[1, \frac{\pi_2^*\left(\beta^* \mid \alpha^{(j)}, \sigma^{(j-1)}, \text{data}\right)}{\pi_2^*(\beta^{(j-1)} \mid \alpha^{(j)}, \sigma^{(j-1)}, \text{data})}\right]$$

and

$$\eta_\sigma = \left[1, \frac{\pi_3^*\left(\sigma^* \mid \alpha^{(j)}, \beta^{(j)}, \text{data}\right)}{\pi_3^*(\sigma^{(j-1)} \mid \alpha^{(j)}, \beta^{(j)}, \text{data})}\right].$$

iii. Generate a u_1, u_2 and u_3 from a Uniform (0,1) distribution.

iv. If $u_1 < \eta_\alpha$, accept the proposal and set $\alpha^{(j)} = \alpha^*$, else set $\alpha^{(j)} = \alpha^{(j-1)}$.

v. If $u_2 < \eta_\beta$, accept the proposal and set $\beta^{(j)} = \beta^*$, else set $\beta^{(j)} = \beta^{(j-1)}$.

vi. If $u_3 < \eta_\sigma$, accept the proposal and set $\sigma^{(j)} = \sigma^*$, else set $\sigma^{(j)} = \sigma^{(j-1)}$.

4) Compute the reliability function, hazard function as

$$S^j(t) = 1 - \left(1 - e^{-\alpha^{(j)}}\right)^{-1}\left[1 - e^{-\alpha^{(j)}\left(1-e^{-\frac{t^2}{2(\sigma^{(j)})^2}}\right)^{\beta^{(j)}}}\right], t \geq 0$$

and

$$h^j(t) = \frac{\alpha^{(j)}\beta^{(j)}(\sigma^{(j)})^{-2} t\, e^{-\frac{t^2}{2(\sigma^{(j)})^2}}\left[1 - e^{-\frac{t^2}{2(\sigma^{(j)})^2}}\right]^{\beta^{(j)}-1} e^{-\alpha^{(j)}\left(1-e^{-\frac{t^2}{2(\sigma^{(j)})^2}}\right)^{\beta^{(j)}}}}{\left(1 - e^{-\alpha^{(j)}}\right) - \left[1 - e^{-\alpha^{(j)}\left(1-e^{-\frac{t^2}{2(\sigma^{(j)})^2}}\right)^{\beta^{(j)}}}\right]}, t \geq 0.$$

5) Set $j = j + 1$.

6) Repeat Steps (3) - (5), N times.

7) The first M simulated variants are deleted to ensure convergence and remove the affection of initial value selection. Then the chosen sample are $\alpha^{(j)}$, $\beta^{(j)}, \sigma^{(j)}, S^j(t)$ and $h^j(t)$, $j = M + 1, \ldots .. N$, for suffciently large N, forms an approximate posterior sample which can be used to develop the Bayes estimates of

$$\hat{\alpha}_M = \frac{1}{N-M}\sum_{j=M+1}^{N} \alpha^{(j)},$$

$$\hat{\beta}_M = \frac{1}{N-M}\sum_{j=M+1}^{N} \beta^{(j)},$$

$$\hat{\sigma}_M = \frac{1}{N-M}\sum_{j=M+1}^{N} \sigma^{(j)},$$

$$\hat{S}(t) = \frac{1}{N-M}\sum_{j=M+1}^{N} S^j(t)$$

and

$$\hat{h}(t) = \frac{1}{N-M}\sum_{j=M+1}^{N} h^{j}(t)\,,$$

where, ($N = 1000$) is the burn-in-period of Markov Chain.

8) To compute the HPD interval of $\alpha, \beta, \sigma, S(t)$ and $h(t)$, order the MCMC sample of $\alpha^{(j)}$, $\beta^{(j)}$, $\sigma^{(j)}$, $S^{j}(t)$ and $h^{j}(t)$ order

$\big(\alpha_1, \alpha_2, \dots, \alpha_N\ as\ \alpha_{[1]}, \alpha_{[2]}, \dots, \alpha_{[N]}, \beta_1, \beta_2, \dots, \beta_N\ as\ \beta_{[1]}, \beta_{[2]}, \dots, \beta_{[N]},\ \sigma_1, \sigma_2, \dots, \sigma_N\ as\ \sigma_{[1]}, \sigma_{[2]},$
$\dots, \sigma_{[N]},\ S_1, S_2, \dots, S_N\ as\ S_{[1]}, S_{[2]}, \dots, S_{[N]},\ h_1, h_2, \dots, h_N\ as\ h_{[1]}, h_{[2]}, \dots, h_{[N]}\big)$

Then construct all $100(1-\nu)$ % credible intervals of

$\Big(\big(\alpha_{(1)}, \alpha_{[N(1-\nu)+1]}\big), \dots, \big(\alpha_{[N\nu]}, \alpha_{[N]}\big)\Big)$ $\Big(\big(\beta_{(1)}, \beta_{[N(1-\nu)+1]}\big), \dots, \big(\beta_{[N\nu]}, \beta_{[N]}\big)\Big)$,
$\Big(\big(\sigma_{(1)}, \sigma_{[N(1-\nu)+1]}\big), \dots, \big(\sigma_{[N\nu]}, \sigma_{[N]}\big)\Big)$, $\Big(\big(S_{(1)}, S_{[N(1-\nu)+1]}\big), \dots, \big(S_{[N\nu]}, S_{[N]}\big)\Big)$ and
$\Big(\big(h_{(1)}, h_{[N(1-\nu)+1]}\big), \dots, \big(h_{[N\nu]}, h_{[N]}\big)\Big)$.

5. SIMULATION STUDY

Compared to parameter estimators and certain lifetime parameters, the reliability function and hazard function of the TW-R distribution. For each simulation, Monte Carlo simulations were carried out using 1000 sample progressively type-II samples. To produce progressively type-II censored samples from TW-R distribution, we used the algorithm [24] with the parameters $\alpha = 0.5, \beta = 1.5$ and $\sigma = 0.5$. The Bayes estimates of unknown quantities, on the basis of 1000 samples, are derived for two squared error loss functions (SEL) and (LINEX). The mean square error (MES), which was calculated to $\varphi = \alpha, \beta, \sigma, S(t)$ and $h(t)$ were considered to be $MSE = \frac{1}{M}\sum_{i=1}^{M}(\hat{\varphi} - \varphi)^2$ in the case of $\alpha, \beta, \sigma, S(t)$ and $h(t)$. We also compare the CIs from asymptotic MLE and MCMC distributions, as well as CRIs. For comparison, the average Credible Interval Length (ACL) and coverage proportions (CP) were applied. The real value was computed within the interval and length of the CI provided for each sample. The 95 percent confidence intervals were computed. This procedure was repeated a thousand times. The estimated likelihood coverage of 1000 true values CIs was determined, and the estimated CI width was derived as the total of all lengths separated by 1000. For every range. For all range, the results for the approximately estimates and MSE are shown in Tables 4, 5, 6, 7 and 8, and 95% CI of ACL and CP is shown in Tables 9 and 10.

Table 1: MSE of ML and Bayes MCMC estimates under SEL and LINEX loss function for the parameter α with $\alpha_0 = 0.5$.

(n,m)	CS	MLE	SEL	LINEX		
				$c_1 = -2.0$	$c_2 = 2.0$	$c_3 = 0.0001$
(30, 20)	I	0.4646 (0.0013)	0.4648 (0.0013)	0.4648 (0.0013)	0.4648 (0.0013)	0.4648 (0.0013)
	II	0.4672 (0.0011)	0.4671 (0.0011)	0.4671 (0.0011)	0.4671 (0.0011)	0.4671 (0.0011)
	III	0.4707 (0.0009)	0.4708 (0.0009)	0.4708 (0.4708)	0.4708 (0.4708)	0.4708 (0.0009)
(40, 2	I	1.2903 (0.1054)	1.2899 (0.1048)	1.2899 (0.1048)	1.2899 (0.1048)	1.2899 (0.1048)
	II	1.1422	1.1423	1.1423	1.1423	1.1423

		(0.155)	(0.1548)	(0.1548)	(0.1548)	(0.1548)
	III	0.9618 (0.3183)	0.9612 (0.3185)	0.9612 (0.3185)	0.9612 (0.3185)	0.9612 (0.3185)
$(60, 40)$	I	1.2426 (0.1073)	1.2431 (0.1073)	1.2431 (0.1073)	1.2431 (0.1073)	1.2431 (0.1073)
	II	0.4666 (0.0011)	0.4665 (0.0011)	0.4665 (0.0011)	0.4665 (0.0011)	0.4665 (0.0011)
	III	0.9615 (0.3073)	0.9617 (0.307)	0.9617 (0.307)	0.9617 (0.307)	0.9617 (0.307)

Table 2: MSE of ML and Bayes MCMC estimates under SEL and LINEX loss function for the parameter β with $\beta_0 = 1.5$.

(n, m)	CS	MLE	SEL	LINEX		
				$c_1 = -2.0$	$c_2 = 2.0$	$c_3 = 0.0001$
$(30, 20)$	I	1.3119 (0.1086)	1.3122 (0.1088)	1.3122 (0.1088)	1.3122 (0.1088)	1.3122 (0.1088)
	II	1.2128 (0.1673)	1.2129 (0.1674)	1.2129 (0.1674)	1.2129 (0.1674)	1.2129 (0.1674)
	III	0.9835 (0.2993)	0.9836 (0.2988)	0.9836 (0.2988)	0.9836 (0.2988)	0.9836 (0.2988)
$(40, 20)$	I	1.2903 (0.1054)	1.2899 (0.1048)	1.2899 (0.1048)	1.2899 (0.1048)	1.2899 (0.1048)
	II	1.1422 (0.155)	1.1423 (0.1548)	1.1423 (0.1548)	1.1423 (0.1548)	1.1423 (0.1548)
	III	0.9618 (0.3183)	0.9612 (0.3185)	0.9612 (0.3185)	0.9612 (0.3185)	0.9612 (0.3185)
$(60, 40)$	I	1.2426 (0.1073)	1.2431 (0.1073)	1.2431 (0.1073)	1.2431 (0.1073)	1.2431 (0.1073)
	II	0.4666 (0.0011)	0.4665 (0.0011)	0.4665 (0.0011)	0.4665 (0.0011)	0.4665 (0.0011)
	III	0.9615 (0.3073)	0.9617 (0.307)	0.9617 (0.307)	0.9617 (0.307)	0.9617 (0.307)

Table 3: MSE of ML and Bayes MCMC estimates under SEL and LINEX loss function for the parameter σ with $\sigma_0 = 0.5$.

(n, m)	CS	MLE	SEL	LINEX		
				$c_1 = -2.0$	$c_2 = 2.0$	$c_3 = 0.0001$
$(30, 20)$	I	0.5815 (0.0076)	0.5817 (0.0076)	0.5817 (0.0076)	0.5817 (0.0076)	0.5817 (0.0076)
	II	0.6112 (0.0141)	0.6109 (0.014)	0.6109 (0.014)	0.6109 (0.014)	0.6109 (0.014)
	III	0.7097 (0.0477)	0.7097 (0.0477)	0.7097 (0.0477)	0.7097 (0.0477)	0.7097 (0.0477)
$(40, 20)$	I	0.5796 (0.0074)	0.5794 (0.0074)	0.5794 (0.0074)	0.5794 (0.0074)	0.5794 (0.0074)
	II	0.6239 (0.0169)	0.6238 (0.0169)	0.6238 (0.0169)	0.6238 (0.0169)	0.6238 (0.0169)
	III	0.7407 (0.0624)	0.7407 (0.0624)	0.7407 (0.0624)	0.7407 (0.0624)	0.7407 (0.7407)
$(60, 40)$	I	0.5848 (0.0079)	0.5848 (0.0079)	0.5848 (0.0079)	0.5848 (0.0079)	0.5848 (0.0079)
	II	0.6219 (0.0155)	0.622 (0.0156)	0.622 (0.0156)	0.622 (0.0156)	0.622 (0.0156)
	III	0.7114 (0.0466)	0.7115 (0.0467)	0.7115 (0.0467)	0.7115 (0.0467)	0.7115 (0.0467)

Table 4: MSE of ML and Bayes MCMC estimates under SEL and LINEX loss function for the parameter for the parameter $S(t)$ with $t = 0.4$.

(n, m)	CS	MLE	SEL	LINEX		
				$c_1 = -2.0$	$c_2 = 2.0$	$c_3 = 0.0001$
$(30, 20)$	I	**0.8314 (0.0028)**	**0.8315 (0.0028)**	**0.8315 (0.0028)**	**0.8315 (0.0028)**	**0.8315 (0.0028)**
	II	**0.8221 (0.0025)**	**0.822 (0.0025)**	**0.822 (0.0025)**	**0.822 (0.0025)**	**0.822 (0.0025)**
	III	**0.8068 (0.0022)**	**0.8069 (0.0022)**	**0.8069 (0.0022)**	**0.8069 (0.0022)**	**0.8069 (0.0022)**
$(40, 20)$	I	**0.8264 (0.0022)**	**0.8262 (0.0022)**	**0.8262 (0.0022)**	**0.8262 (0.0022)**	**0.8262 (0.0022)**
	II	**0.827 (0.0015)**	**0.8268 (0.0015)**	**0.8268 (0.0015)**	**0.8268 (0.0015)**	**0.8268 (0.0015)**
	III	**0.814 (0.0014)**	**0.8139 (0.0014)**	**0.8139 (0.0014)**	**0.8139 (0.0014)**	**0.8139 (0.0014)**
$(60, 40)$	I	**0.8207 (0.0015)**	**0.8208 (0.0015)**	**0.8208 (0.0015)**	**0.8208 (0.0015)**	**0.8208 (0.0015)**
	II	**0.8165 (0.0014)**	**0.8166 (0.0013)**	**0.8166 (0.0013)**	**0.8166 (0.0013)**	**0.8166 (0.0013)**
	III	**0.8046 (0.0014)**	**0.8047 (0.0014)**	**0.8047 (0.0014)**	**0.8047 (0.0014)**	**0.8047 (0.0014)**

Table 5: MSE of ML and Bayes MCMC estimates under SEL and LINEX loss function for the parameter $h(t)$ $with$ $t = 0.4$.

(n, m)	CS	MLE	SEL	LINEX		
				$c_1 = -2.0$	$c_2 = 2.0$	$c_3 = 0.0001$
$(30, 20)$	I	5.1165 (2.3406)	5.111 (2.3384)	5.1116 (2.3381)	5.1116 (2.3387)	5.111 (2.3384)
	II	5.1637 (0.0025)	5.1696 (1.8819)	5.1702 (1.8812)	5.1702 (1.8826)	5.1696 (1.8819)
	III	5.0452 (1.5141)	5.0426 (1.5097)	5.0432 (1.5089)	5.0432 (1.5105)	5.0426 (1.5097)
$(40, 20)$	I	5.2658 (1.6872)	5.2712 (1.6807)	5.2719 (1.6804)	5.2719 (1.6811)	5.2712 (1.6807)
	II	4.9942 (1.6112)	4.9985 (1.6024)	4.9992 (1.6017)	4.9992 (1.6031)	4.9985 (1.6024)
	III	4.7569 (1.7696)	4.7596 (1.7544)	4.7602 (1.7533)	4.7602 (1.7555)	4.7596 (1.7544)
$(60, 40)$	I	5.3913 (1.0971)	5.3897 (1.1034)	5.39 (1.1032)	5.39 (1.1037)	5.3897 (1.1034)
	II	5.2684 (1.0501)	5.2649 (1.0462)	5.2652 (1.0459)	5.2652 (1.0466)	5.2649 (1.0462)
	III	5.0944 (1.0344)	5.0904 (1.0377)	5.0907 (1.0373)	5.0907 (1.0381)	5.0904 (1.0377)

Table 6: ACL and CP of 95% CIs for the parameters α, β and σ

(n, m)	CS	α		β		σ	
		MLE	MCMC	MLE	MCMC	MLE	MCMC
$(30, 20)$	I	0.3639 (0.961)	0.0025 (0.966)	1.4049 (0.9408)	0.009 (0.9641)	-0.3099 (0.9702)	0.0024 (0.9284)
	II	0.3652 (0.9421)	0.0024 (0.9277)	1.3139 (0.936)	0.009 (0.9719)	-0.3348 (0.9606)	0.0027 (0.9413)
	III	0.367 (0.9747)	0.0025 (0.9287)	1.0805 (0.9299)	0.0071 (0.9448)	-0.4337 (0.9598)	0.0037 (0.9481)
$(40, 20)$	I	0.3643 (0.9723)	0.0026 (0.9575)	1.2942 (0.9565)	0.0089 (0.9251)	-0.3082 (0.9721)	0.0024 (0.9694)
	II	0.3653 (0.9485)	0.0025 (0.9464)	1.1654 (0.9589)	0.008 (0.9582)	-0.3461 (0.9503)	0.0029 (0.9309)
	III	0.3672 (0.9269)	0.0025 (0.9628)	0.999 (0.9362)	0.0066 (0.9269)	-0.4587 (0.9658)	0.0043 (0.9644)
$(60, 40)$	I	0.2575 (0.9291)	0.0017 (0.9702)	0.9454 (0.9732)	0.0066 (0.9486)	-0.3277 (0.9309)	0.0017 (0.9302)
	II	0.258 (0.9737)	0.0018 (0.9253)	0.8505 (0.9452)	0.0056 (0.9408)	-0.3664 (0.9618)	0.002 (0.9747)
	III	0.2596 (0.9335)	0.0018 (0.9646)	0.7422 (0.9309)	0.0049 (0.9256)	-0.4712 (0.9601)	0.0026 (0.9372)

Table 7: ACL and CP of 95% CIs for the parameters $S(t)$ and $h(t)$.

		$S(t)$		$h(t)$	
(n,m)	CS	MLE	MCMC	MLE	MCMC
$(30,20)$	I	0.2338 (0.9648)	0.0027 (0.9563)	6.6623 (0.9708)	0.0824 (0.9746)
	II	0.2224 (0.9621)	0.0027 (0.9647)	6.141 (0.9631)	0.0814 (0.9266)
	III	0.2295 (0.9643)	0.0029 (0.9475)	5.6449 (0.9355)	0.0797 (0.9432)
$(40,20)$	I	0.224 (0.9398)	0.0026 (0.9744)	6.5431 (0.9434)	0.081 (0.9479)
	II	0.1953 (0.9575)	0.0028 (0.963)	5.5654 (0.9448)	0.0844 (0.9641)
	III	0.1977 (0.9277)	0.0029 (0.9712)	4.9885 (0.9639)	0.0783 (0.9522)
$(60,40)$	I	0.1754 (0.94)	0.002 (0.9496)	4.9231 (0.9298)	0.0601 (0.9648)
	II	0.1605 (0.9683)	0.002 (0.9642)	4.3649 (0.9478)	0.0589 (0.9711)
	III	0.1639 (0.9725)	0.002 (0.9548)	3.9953 (0.9457)	0.0532 (0.9436)

From the results, we observe the following:

1) It is observed that from Tables 4, 5, 6, 7 and 8, as sample size increases, the MSEs decrease and Bayes estimates have the smallest MSEs for $\alpha, \beta, \sigma, S(t)$ and $h(t)$. Hence, Bayes estimates perform better than the MLEs methods in all cases considered.

2) The Bayes estimates are better in terms $\alpha, \beta, \sigma, S(t)$ and $h(t)$ of having reduced MSEs .

3) The LINEX estimates with $c = 2.0$ are better estimates for smaller MSEs with $c = -2.0$ and 0.0001.

4) In terms of MSEs for samples *n* fixed values and failure time sizes *m*, Scheme I performs better than Schemes II and III.

5) Tables 9 and 10, The MCMC, It can be seen that, the CRIs give more accurate results than the ACIs, for different sample sizes, observed failures and schemes.

6. APPLICATIONS TO REAL DATA

We are presenting numerical results for estimating TW-R model parameters under progressive type II censoring with real data sets [3]. The 63 aircraft Windshield failed times, the windshield is a complicated component of a big aircraft, with 153 observations, eighty-eight of which are classified as failed windshields, and the other sixty-five are service periods for windshields that were not failed when observation took place. The data is

Table 8: Progressively censored sample based on data of 63 aircraft Windshield failed times.

X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i
0.046	**7**	**1.719**	**0**	**1.963**	**0**	**3.304**	**0**	**4.628**	**0**	**1.085**
1.436	**6**	**2.717**	**0**	**2.950**	**0**	**0.996**	**0**	**1.244**	**0**	**2.163**
2.592	**21**	**1.794**	**0**	**0.622**	**0**	**2.117**	**0**	**2.435**	**0**	**3.665**
0.140	**10**	**2.819**	**0**	**1.978**	**0**	**3.483**	**0**	**4.806**	**0**	**1.092**
1.492	**0**	**0.313**	**0**	**3.003**	**0**	**1.003**	**0**	**1.249**	**0**	**2.183**
2.600	**0**	**1.915**	**0**	**0.900**	**0**	**2.137**	**0**	**2.464**	**0**	**3.695**
0.150	**0**	**2.820**	**0**	**2.053**	**0**	**3.500**	**0**	**4.881**	**0**	**1.152**
1.580	**0**	**0.389**	**0**	**3.102**	**0**	**1.010**	**0**	**1.262**	**0**	**2.240**
2.670	**0**	**0.280**	**0**	**0.952**	**0**	**2.141**	**0**	**2.543**	**0**	**1.183**
0.248	**0**	**1.920**	**0**	**2.065**	**0**	**3.622**	**0**	**5.140**	**0**	**2.341**
2.878	**0**	**0.487**	**0**	**4.015**	**0**					

The real data was discussed on the basis of progressive censoring of type-II data for the Truncated Weibull-Rayleigh distribution (TW-R). For data, where $K - S = 0.108456$ we measured the distance of

Kolmogorov-Smirnov $(K-S)$ between the empirical function and the fitting distributions. In cases of complete real data censoring, the convergence of the MCMC estimates for α, β and σ can be shown in Table 2. Table 3 list 95% probability intervals, reliability function and hazard function, show the results for Bayes estimates of both SEL and LINEX with different values of the LINEX loss function for the α, β and σ parameters, as well as for $S(t=0.5)$ and $h(t=0.5)$ parameters in Table 2.

Table 9: MLE and Bayes MCMC estimates under SEL and LINEX for Real Data

Parameters	MLE	SEL	LINEX		
			$c_1 = -0.2$	$c_2 = 0.2$	$c_3 = 0.0001$
α	795.675	795.675	795.675	795.675	795.675
β	0.814316	0.814931	0.814931	0.81493	0.814931
σ	106.086	104.388	104.746	104.138	104.388
$S(t=0.5)$	0.929137	0.927802	0.927802	0.927801	0.927802
$h(t=0.5)$	1.91329	2.1067	2.1067	2.1067	2.1067

Table 10: 95% CIs of $\alpha, \beta, \sigma, S(t)$ and $h(t)$ for Real Data

Parameters	MCMC	ACIs
α	(795.675,795.675)	(795.675,895.675)
β	(0.812876,0.816798)	(0.76316,0.965472)
σ	(103.492,105.691)	(83.38,128.792)
$S(t=0.5)$	(0.926386,0.930212)	(0.901143,0.997132)
$h(t=0.5)$	(1.50089, 2.0205)	(1.29666,2.58002)

For c approaching zero, the LINEX loss function is symmetrical, and hence behaves identically to the squared error loss function. We further found that the resulting $c=0.0001$ estimates are roughly equivalent to the corresponding Bayes estimates of the squared error. Finally, it is clear that the confidence intervals lengths of the Bayes estimators for α, β and σ are smaller than their MLEs.

7. CONCLUSION

This paper is designed to improve various methods of estimating and constructing confidence intervals for the parameters and the reliability and hazard function of the Truncated Weibull Rayleigh (TW-R) distribution. The MLEs of the unknown parameters are obtained using asymptotic distributions to suggest different confidence intervals. The unknown parameters are also proposed in the Bayesian estimates. It is noted that Bayes estimators cannot be obtained explicitly and can be obtained with the help of numerical integration. We used MCMC and it is noted that the Bayesian results in practical situations. In addition, loss functions of the Bayes estimates were obtained. The numerical illustration was used to illustrate the theoretical results. In a simulation studies the performance of the proposed methods for various sample sizes $(n; m)$ and various CSs (I,II,III) was examined. A simulation study is performed to assess the quality of the proposed estimators and this study showed that the Bayesian methods have good performance in all different cases. Hence, it is recommended that depending on the Bayesian results in practical situations.

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